International Journal of Applied Mathematics in Control Engineering 2 (2018) 111-118

Contents lists available at YXpublications

International Journal of Applied Mathematics in Control Engineering

Journal homepage: http://www.ijamce.com

Finite-Time Stability of Fractional Order Neural Networks with Proportional Delays

Shengnan Wen*, Deyou Liu, Xiaoyuan Wang

School of Science, Yanshan University, Qinhuangdao, 066004, China;

ARTICLE INFO

Article history:
Received 28 March 2018
Accepted 28 June 2018
Available online 25 December 2018

Keywords:
Fractional-order
Neural Networks
Proportional Delays
Finite-time Stability

ABSTRACT

In this paper, the finite time stability of fractional non-autonomous neural networks with heterogeneous proportional delays is studied. According to the principle of homomorphic mapping, the nonlinear mapping is constructed to prove the existence of the unique solution of the system. Using the Leibniz rules comparison on the propagation of the fractional order differential technology, the sequence based on time-delay neural network model is deduced, be considered to ensure that the fractional order neural network model on the global stability. Two numerical examples are given to verify the validity of the results.

1. Introduction

Fractional calculus differentiates the differential and integral from the general integer order to any real number, as a branch of calculus, which has been developing for more than 300 years (Podlubny, 1999, Luo et al., 2018, Shi et al., 2018). Because the fractional order calculus compared with classical calculus, has a special complexity, so at the beginning of the study, although there are a lot of scholars devoted to them, but in a very long period of time is just the problem in the field of fractional order calculus as pure mathematics (Li and Deng, 2007, Cao and Zhou, 1998, Gupta et al., 2003, Yu et al., 2013), and can't apply it to the actual background and physical significance. In 1974, k.b. idham and j.s. panier published the first treatise on fractions in New York press, prompting people to start paying attention to fractions. In 1993, Samko created <Fractional Integrals and Derivatives: Theory and Applications>. This book systematically and comprehensively elaborates some related properties of fractional derivative and integral and related applications (Samko et al., 1993, Wu et al., 2018).

Fractional order calculus has a unique advantage, the theory of differential model overcame the classical integer order differential model results match well with the experimental results of defects, and simply rely on fewer parameters can well describe the physical memory and genetic properties (Shi and Wang, 2011, Koeller, 1984, Yu et al., 2014). The concept of artificial neural network was first proposed by psychologist McCulloch and mathematician Pitts in 1943. In 1982, the United States biophysicist J. Hopfield named after the name of Hopfield neural network model, and studied it's stability, at the same time also gives a stability criterion to determine the neural network equilibrium (Mandelbort, 1982).

In the 1880 s, D.Rumelhart et al. developed to solve the multilayer neural network weights are revised BP algorithm, and shows that the multilayer neural network has strong learning ability, can solve many problems in real life, to promote the sustainable development of the neural network study (Wang et al., 2015). In 1998, the stage for the first time, scientists such as cellular neural network system consists of an integer order differential across to the fractional order calculus (Diethelm, 2010), on this basis, the analysis of system dynamic properties. In the real world, the mathematical model of neural network can appear not only systems with time-delay and parameter selection problem such as in stability, also will be affected by such as all kinds of uncertainty, complexity of model itself, such as impact (Petras, 2011, Arena et al., 2000, Yan et al., 2010). The fractional order neural network as the main object of study, analysis of system stability, pattern recognition, the respect such as projection, optimization problems, of both theoretical results and practical application results are with very high value of exploration.

As one of the basic characteristics of the differential system, stability can guarantee the normal operation of the control system. The earliest literature on the stability of fractional differential system is shown in (Matignon, 1996). In this paper, a class of fractional order Caputo is studied. With the development of fractional differential equations, how to determine the stability of

* Corresponding author.
E-mail addresses: shengnan.wen@foxmail.com(S. Wen)

fractional differential equations gradually becomes an important subject (Li et al., 2009, Delavari, et al., 2012). In the real world, time-lag is prevalent in many fields, such as economy, electromechanical engineering, biology, finance, etc. (Boroomand and Menhaj, 2009, Corduneanu, 1971, Li et al., 2009). To a certain extent, time delay affects the working status of all systems, and becomes a major source of unstable factors and poor performance of the system (Ren and Zhang, 2007, Ma and Chen 2015, Luo, 2007). And the dynamic behavior of the neural network system which we face would mostly appear in the limited time range. So the system's finite time stability dynamics behavior is more popular. However, there is still less literature on the limited time stability of neural network. In Aghababa (Aghababa et al., 2011), studied the limited time synchronization of two different chaotic systems by using synovial control technology. In Wu (Wu et al., 2013) studied the finite time stability analysis of complex neural networks with time delay using Laplace transform, mittag-leffler function and generalized Gronwall inequality.

In recent years, a special type of time-varying delay is introduced in many neural network models. What differs from the traditional type is that it appears in the domain of the Internet is unbounded time change. In the neural network model with proportional delays, the system is determined by $x(t)$ and $x(qt)$ together in the dynamic state of time t, $0<q<1$ is a constant, indicating the time ratio between the current state and the historical state.

In Yang (Yang and Cao 2013), based on the theory of differential equation and the M - matrices, is derived from the perspective of matrix inequality new explicit conditions, The finite time stability of a class of non-autonomous neural networks has been studied. In Hien (Hien and Son, 2015) and Liu (Liu, 2017) is by means of differential inequality technique, discussed two kinds of heterogeneous proportional delays and oscillation leakage coefficient of non-autonomous cellular neural networks limited time stability problems.

The rest of the paper is organized as follows. Some necessary definitions and lemmas are recalls in Section 2.The obtained results on finite-time stability of non-autonomous FONNs with heterogeneous proportional delays are presented in Sect.3. Two examples with numerical simulations are givens in Section 4.

Throughout the paper, $\underline{n}\overset{\Delta}{=}\{1,2,\dots,n\}$ for a positive integer n, For any two vectors $u=(u_i)\in R^n$ and $v=(v_i)\in R^n$, if $u_i\ge v_i$, then $u\ge v$; if $u_i>v_i$, then $u\gg v$. For a vector $\xi\in R^n$, $\xi\gg 0$, we denote $\xi^u=\max_{i\in\underline{n}}\xi_i$ and $\xi_l=\min_{i\in\underline{n}}\xi_i$.

2. Preliminaries

Definition 2.1 (Podlubny, 1999) The fractional integral is defined as

$$ {}_a^{RL}D_t^{-p}f(t)=\frac{1}{\Gamma(p)}\int_a^t(t-\tau)^{p-1}f(\tau)\mathrm{d}\tau $$

where $p>0$, $\Gamma(\cdot)$ is the Gamma function,

$$ \Gamma(z)=\int_0^{+\infty}t^{z-1}e^{-t}\mathrm{d}t,(\Re e(z)>0). $$

Definition 2.2 (Podlubny, 1999) The Riemann-Liouville derivative of fractional order p of function $f(t)$

$$ {}_a^{RL}D_t^{p}f(t)=\frac{1}{\Gamma(n-p)}\frac{d^n}{dt^n}\int_a^t\frac{f(\tau)}{(t-\tau)^{p-n+1}}\mathrm{d}\tau, $$

where $n-1\le p<n$.

Definition 2.3 (Podlubny, 1999) The Caputo derivative of fractional order p of function $f(t)$

$$ {}_a^{C}D_t^{p}f(t)=\frac{d^pf(t)}{d(t-a)^p}=\frac{1}{\Gamma(n-p)}\int_a^t(t-\tau)^{n-p-1}f^{(n)}(\tau)\mathrm{d}\tau $$

where $p>0$, $n-1\le p<n$.

Definition 2.4 (Ren and Zhang, 2007) Topological space X and Y, if the mapping $F:X\to Y$ satisfies:

1) F is bijection;

2) F is continuous;

3) F have continuous inverse functions F^{-1};

then F is a homeomorphism mapping.

Definition 2.5 (Ren and Zhang, 2007) Assume that real matrix $B=(b_{ij})_{n\times n}$ satisfies that if $i=j$, $b_{ij}>0$; when $i\ne j$, $b_{ij}\le 0$, $i,j=1,2,\dots,n.$ And all of the matrices are positive, then B is a M -matrix.

Lemma 2.1 (Ren and Zhang, 2007) The following conditions are equivalent to matrix $B\in R^{n\times n}$:

1) B is a M - matrix;

2) The real part of all the characteristic roots of B is positive;

3) B is reversible and $B^{-1}\ge 0$, B^{-1} non-negative;

4) for a vector $\alpha>0$, $B\alpha>0$;

5)There is a positive diagonal matrix Q, making $QB+B^TQ$ the positive definite.

Lemma 2.2 (Ren and Zhang, 2007) $f(x)$ is continuous, and

1) $f(x)$ is an injective on R^n;

2) $\lim\limits_{\|x\|\to+\infty}\|f(x)\|\to+\infty$;

then, $f(x)$ is homeomorphism mapping on R^n.

Lemma 2.3 (Cauchy-Schwartz inequality) For any two real numbers $u_i,v_i,i=1,2,\dots,n$

$$ \left(\sum_{i=1}^n u_iv_i\right)^2\le\left(\sum_{i=1}^n u_i^2\right)\left(\sum_{i=1}^n v_i^2\right), $$

when $u_i=\lambda v_i,i=1,2,\dots,n$,the inequality is medium, where λ is a constant.

Lemma 2.4 (Podlubny, 1999) For a function $f(\cdot)\in C^1[0,\infty)$ and a scalar $0<p<1$, if the function $\varphi(\cdot)$ and all its derivatives are continuous on $[0,t]$, $t>0$, then the following Leibniz rule for fractional differentiation holds

$$ {}_0^{RL}D_t^{p}(\varphi(t)f(t))=\sum_{k=0}^n\binom{p}{k}\frac{d^k\varphi(t)}{dt^k}\,{}_0^{RL}D_t^{p-k}f(t)-R_n^p(t), $$

where n is an integer such that $n\ge p+1$,

$$ \binom{p}{k}=\frac{\Gamma(p+1)}{k!\Gamma(p-k+1)}, $$

$$R_n^p(t)=\frac{(-1)^n(t-p)^{n-p+1}}{n!\Gamma(-p)}\int_0^1\int_0^1 F_p(t,u,v)dudv,$$

$$F_p(t,u,v)=f(vt)\varphi^{(n+1)}(t(u+v-uv)).$$

3. Main Results

In this section, two sufficient conditions are derived for a class of fractional order neural networks with proportional delays.

Consider the following fractional non-autonomous time-varying delay systems:

$$\begin{cases} {}_{t_0}^{C}D_t^p x_i(t)=-d_i(t)x_i(t)+\sum_{j=1}^{n}a_{ij}(t)f_j(x_j(t)) \\ \qquad +\sum_{j=1}^{n}b_{ij}g_j(x_j(q_{ij}t))+I_i(t), \quad t>0 \qquad (1) \\ x_i(0)=x_i^0, i\in 1,2,\ldots,n. \end{cases}$$

n is the number of units in a neural networks, $x_i(t)\in R^n$ is the state variable of i th neuron at time t, $I_i(t)$ is the external input, $d_i(t)$ is the self-inhibition rate at which the i th neuron will reset its potential to the resting state in isolation when disconnected from the network and external input; $A=(a_{ij})_{n\times n}$, $B=(b_{ij})_{n\times n}$ are time-varying connection weights, $f_j(x_j(t))$ and $g_j(x_j(t))$ are the neuron activation functions, q_{ij} represents the connection proportional delay from i th neuron to j th, where

$$0<q_{ij}\le 1, \quad q_{ij}t=t-(1-q_{ij})t,$$

$(1-q_{ij})t$ is signal transmission delay function, when $t\to+\infty$, $(1-q_{ij})t\to+\infty$, the time-delay term is an unbounded time delay function of model (1). $J=[t_0,t_0+T], J\subset R$ (T is a positive number or $+\infty$).

Assume that $x(t)=(x_1(t),x_2(t),\ldots,x_n(t))^T$ and $y(t)=(y_1(t), y_2(t),\ldots,y_n(t))^T$ are any two solutions of (1) with different initial functions $\varphi\in C$ and $\phi\in C$, let $z(t)=y(t)-x(t)=(z_1(t),z_2(t),\ldots,z_n(t))^T$, $t>0$, $i\in \underline{n}$, $\psi=\phi-\varphi$. One has the error system

$$\begin{cases} {}_{t_0}^{C}D_t^p z_i(t)=-d_iz_i(t)+\sum_{j=1}^{n}a_{ij}\left(f_j(y_j(t))-f_j(x_j(t))\right) \\ \qquad +\sum_{j=1}^{n}b_{ij}\left(g_j(y_j(q_{ij}t))-g_j(x_j(q_{ij}t))\right), \qquad (2) \\ z(t_0+\theta)=\psi(\theta), \quad \theta>0, \quad i\in\underline{n}. \end{cases}$$

where $\psi\in C$ is the initial function of system (2), define the norm

$$\|\psi\|=\sup_{\theta\in(0,T]}\|\psi(\theta)\|.$$

Definition 3.1. System (1) is said to be finite-time stable w.r.t. $\{\delta,\varepsilon,t_0,J\}$, if and only if $\|z_0\|<\delta$ implies $\|z(t)\|<\varepsilon$, $\forall t\in J=[t_0,t_0+T]$, where $z(t_0)=z_0$, t_0 is the initial time of observation, δ,ε,T are real positive numbers and $\delta<\varepsilon$.

In order to obtain main results, make the following assumptions:

(N1) d_i is continuous defined on R, for any two scalars u,v

$$\frac{d_i(u)-d_i(v)}{u-v}\ge D_i, \quad i=1,2,\ldots n.$$

(N2)The neuron activation functions $f_j,g_j(j=1,2,\ldots,n)$ are Lipschitz continuous, that is there exist positive constants F_j, $G_j>0$ such that

$$|f_j(u)-f_j(v)|\le F_j|u-v|,$$

$$|g_j(u)-g_j(v)|\le G_j|u-v|, \qquad \forall u,v\in R$$

We denote

$$D=diag(D_1,D_2,\ldots,D_n),$$
$$F=diag(F_1,F_2,\ldots,F_n),$$
$$G=diag(G_1,G_2,\ldots,G_n).$$

In addition, time-varying connection weights $a_{ij}(t), b_{ij}(t)$ and the coefficients $d_i(t)$ and the input vector $I(t)=(I_i(t))$ are assumed to be continuous on R_+.

Assuming that the equilibrium point of the system (1) is solution $x^*=(x_1^*,x_2^*,\ldots,x_n^*)^T$, then

$$-d_ix_i^*+\sum_{j=1}^{n}a_{ij}f_j(x_j^*)+\sum_{j=1}^{n}b_{ij}g_j(x_j^*))+I_i=0 \qquad (3)$$
$$i=1,2,\ldots,n.$$

According to equation (3), the nonlinear mapping is taken

$$F(x)=-d(x)+Af(x)+Bg(x)+I, \qquad (4)$$

The resulting solution $F(x)=0$ is the equilibrium point of system (1).

Assume that $F(x)$ is a homeomorphism mapping defined on R^n, according to **Definition 2.4**, $F(x)$ is a surjection, then, there has a point $x^*=(x_1^*,x_2^*,\ldots,x_n^*)^T$ makes $F(x^*)=0$; $F(x)$ is an injection, so there is a unique x^* makes $F(x^*)=0$. So we know the existence and uniqueness of the system equilibrium point. Then, we turned to prove $F(x)$ to be a homomorphic mapping.

Theorem 3.1. If (N1) and (N2) hold, $D-|A|F-|B|G$ is a M-matrix, the system (1) has a unique balance。

Proof: $F(x)$ is continuous on R^n, we prove it in two parts according to the **Lemma 2.2**.

1) First, let's prove $F(x)$ is a injective.

Assume that $x,y\in R^n$ satisfy $x\neq y, F(x)=F(y)$, then according to assumption (N1) and (N2),

$$
\begin{aligned}
&|F(x)-F(y)| \\
&=\left|-\left(d(x)-d(y)\right)+A(f(x)-f(y))+B(g(x)-g(y))\right| \\
&\geq |d(x)-d(y)|-|A||f(x)-f(y)|-|B||g(x)-g(y)| \\
&\geq (D-|A|F-|B|G)|x-y|.
\end{aligned}
$$

Known that $F(x)=F(y)$, so $|F(x)-F(y)|=0$,

$$(D-|A|F-|B|G)|x-y|\leq 0.$$

According to **Lemma 2.1**, $(D-|A|F-|B|G)^{-1}\geq 0$, then

$$
\begin{aligned}
&(D-|A|F-|B|G)^{-1}\left[(D-|A|F-|B|G)|x-y|\right] \\
&=|x-y|\leq 0.
\end{aligned}
$$

This contradicts what was mentioned above $|x-y|>0$, so $F(x)$ is a injective on R^n。

2) The proof of $\lim\limits_{\|x\|\to+\infty}\|F(x)\|\to+\infty$ is followed.

According to **Lemma 2.1**, there is a positive diagonal matrix $Q=diag(Q_1,Q_2,\ldots,Q_n)$

$$
\begin{aligned}
&Q(D-|A|F-|B|G)+(D-|A|F-|B|G)^T Q \\
&=Q(D-|A|F-|B|G)+(D-|A|F-|B|G)^T Q^T \\
&=Q(D-|A|F-|B|G)+(Q(D-|A|F-|B|G))^T>0,
\end{aligned}
$$

Instead of the positive definite matrix transpose, and also not positive, so

$$Q(D-|A|F-|B|G)|x-y|>0,$$

There must be a minimum $\varepsilon>0$, an unit matrix E_n, make

$$Q(D-|A|F-|B|G)|x-y|\geq \varepsilon E_n>0.$$

Note that $\overline{F}(x)=F(x)-F(x)$, from assumptions (N1) and (N2) and equation (4), one has

$$
\begin{aligned}
&(Q|x|)^T\left|\overline{F}(x)\right| \\
&=(Q|x|)^T\left|-\left(d(x)-d(0)\right)+A(f(x)-f(0))+B(g(x)-g(0))\right| \\
&\geq |x|^T Q\left[|d(x)-d(0)|-|A||f(x)-f(0)|-|B||g(x)-g(0)|\right] \\
&\geq |x|^T\left[Q(D-|A|F-|B|G)\right]|x|\geq \varepsilon\|x\|^2.
\end{aligned} \tag{5}
$$

From **Lemma 2.3**,

$$
\begin{aligned}
&(Q|x|)^T\left|\overline{F}(x)\right| \\
&=|x|^T Q^T\left|\overline{F}(x)\right| \\
&=|x_1|Q_1\left|\overline{F}_1\right|+|x_2|Q_2\left|\overline{F}_2\right|+\cdots+|x_n|Q_n\left|\overline{F}_n\right| \\
&\leq \max\{Q_1,Q_2,\ldots,Q_n\}\left(\left|x_1\overline{F}_1\right|+\left|x_2\overline{F}_2\right|+\cdots+\left|x_n\overline{F}_n\right|\right) \\
&\leq \|Q\|(x_1^2,x_2^2,\ldots,x_n^2)^{1/2}(\overline{F}_1^2+\overline{F}_2^2+\cdots+\overline{F}_n^2)^{1/2} \\
&=\|Q\|\|x\|\left\|\overline{F}(x)\right\|.
\end{aligned} \tag{6}
$$

From equation (5) and (6), one obtains $\|Q\|\|x\|\left\|\overline{F}(x)\right\|\geq\varepsilon\|x\|^2$, that is

$$\left\|\overline{F}(x)\right\|\geq\frac{\varepsilon\|x\|}{\|Q\|}.$$

It means that $\|x\|\to+\infty$, when $\left\|\overline{F}(x)\right\|\to+\infty$, it follows that $\|F(x)\|\to+\infty.$

To sum up, we have the conclusion that the system (1) has a unique equilibrium point x^*.

In order to prove our conclusion, the following conditions are given:

(C1)There is a constant $r>0$ that satisfies the inequality

$$-d_i(t)+\sum_{j=1}^{n}(|a_{ij}(t)||F_j|+|b_{ij}(t)|\frac{|G_j|}{q_{ij}^p})+\frac{(p^2+p^{-1})}{r^p\Gamma(2-p)}\leq 0,\forall i\in \underline{n}.$$

Theorem 3.2 If the Assumptions (N1) and (N2) and the conditions (C1) are all set up, the system (1) is finite-time stable.

Proof: System (1) has a unique solution. Let

$$
\begin{aligned}
x(t)&=(x_1(t),x_2(t),\ldots,x_n(t))^T \\
y(t)&=(y_1(t),y_2(t),\ldots,y_n(t))^T
\end{aligned}
$$

be any two solutions of system (1). Define $z_i(t)=y_i(t)-x_i(t)$, $t>0, i\in\underline{n}.$ $z(0)=x_1^0-x_2^0.$

From system (1), one has

$$
\begin{aligned}
{}_0^C D_t^p z_i(t)&=-d_i z_i(t)+\sum_{j=1}^{n}a_{ij}\left(f_j(y_j(t))-f_j(x_j(t))\right) \\
&+\sum_{j=1}^{n}b_{ij}\left(g_j(y_j(q_{ij}t))-g_j(x_j(q_{ij}t))\right),t>0
\end{aligned} \tag{7}
$$

According to Assumption (N2), hence

$$
\begin{aligned}
&{}_0^C D_t^p|z_i(t)|\leq \operatorname{sgn}(z_i(t))\,{}_0^C D_t^p z_i(t) \\
&\leq -d_i(t)|z_i(t)|+\sum_{j=1}^{n}|a_{ij}(t)|\left|\left(f_j(y_j(t))-f_j(x_j(t))\right)\right| \\
&\quad+\sum_{j=1}^{n}|b_{ij}(t)|\left|\left(g_j(y_j(q_{ij}t))-g_j(x_j(q_{ij}t))\right)\right| \\
&\leq -d_i(t)|z_i(t)|+\sum_{j=1}^{n}|a_{ij}(t)||F_j||z_j(t)| \\
&\quad+\sum_{j=1}^{n}|b_{ij}(t)||G_j||z_j(q_{ij}t)|,\qquad \forall i\in\underline{n},\qquad t>0.
\end{aligned} \tag{8}
$$

Consider the following function

$$V(t)=(t+r)^p\varphi(t),\qquad t\geq 0.$$

We denote,

$$\varphi(t)=|z_i(t)|,\varphi(t)=\sup_{0\leq s\leq t}\varphi(s),\ \hat{V}(t)=\sup_{0\leq s\leq t}V(t).$$

Then we show that

$${}_0^C D_t^p V(t)\leq(t+r)^p\,{}_0^C D_t^p\varphi(t)+\frac{(p^2+p^{-1})\hat{V}(t)}{r^p\Gamma(2-p)},\forall t>0. \tag{9}$$

According to the relationship between Caputo derivative and

Liemann-liouville derivative and **Lemma 2.4**. One has

$$
\begin{aligned}
{}_0^C D_t^p V(t) &= {}_0^{RL} D_t^p V(t) - \frac{V(0)}{\Gamma(1-p)} t^{-p} \\
&= {}_0^{RL} D_t^p (t+r)^p \varphi(t) - \frac{r^p \varphi(0)}{\Gamma(1-p)} t^{-p} \\
&\le (t+r)^p \, {}_0^{RL} D_t^p \varphi(t) - \frac{r^p \varphi(0)}{\Gamma(1-p)} t^{-p} \\
&\quad + p(t+r)^{p-1} \frac{\Gamma(p+1)}{\Gamma(p)} \, {}_0^{RL} D_t^{p-1} \varphi(t) \\
&= (t+r)^p \, {}_0^C D_t^p \varphi(t) + p^2 (t+r)^{p-1} \, {}_0^{RL} D_t^{p-1} \varphi(t) \\
&\quad + \frac{\varphi(0)}{\Gamma(1-p)} \left[\left(\frac{t+r}{t}\right)^p - \left(\frac{r}{t}\right)^p \right] \qquad (10) \\
&\le (t+r)^p \, {}_0^C D_t^p \varphi(t) + \frac{\varphi(0)}{\Gamma(1-p)} \\
&\quad + p^2 (t+r)^{p-1} \, {}_0^{RL} D_t^{p-1} \varphi(t) \\
&\le (t+r)^p \, {}_0^C D_t^p \varphi(t) + \frac{p^{-1} \hat{V}(t)}{r^p \Gamma(2-p)} \\
&\quad + p^2 (t+r)^{p-1} \, {}_0^{RL} D_t^{p-1} \varphi(t)
\end{aligned}
$$

In the formula above,

$$
\begin{aligned}
& p^2 (t+r)^{p-1} \, {}_0^{RL} D_t^{p-1} \varphi(t) \\
&= \frac{p^2}{\Gamma(1-p)} (t+r)^{p-1} \int_0^t (t-s)^{-p} \varphi(s) ds \\
&\le \frac{p^2}{\Gamma(1-p)} (t+r)^{p-1} \varphi(t) \int_0^t (t-s)^{-p} ds \\
&= \frac{p^2}{(1-p)\Gamma(1-p)} \varphi(t) \left(\frac{t}{t+r}\right)^{p-1} \qquad (11) \\
&\le \frac{p^2}{r^p \Gamma(2-p)} \hat{V}(t) \\
& t > 0
\end{aligned}
$$

From (10) and (11),

$$
{}_0^C D_t^p V(t) \le (t+r)^p \, {}_0^C D_t^p \varphi(t) + \frac{(p^2 + p^{-1}) \hat{V}(t)}{r^p \Gamma(2-p)}, \forall t > 0.
$$

From (8), the Caputo derivative of $\varphi(t)$ meets

$$
\begin{aligned}
{}_0^C D_t^p \varphi(t) &\le -d_i(t) |z_i(t)| + \sum_{j=1}^n |a_{ij}(t)| |F_j| |z_j(t)| \\
&\quad + \sum_{j=1}^n |b_{ij}(t)| |G_j| |z_j(q_{ij} t)| \\
&\le -d_i(t) \varphi(t) + \sum_{j=1}^n |a_{ij}(t)| |F_j| \varphi(t) + \sum_{j=1}^n |b_{ij}(t)| |G_j| \varphi(q_{ij} t) \\
&\le -d_i(t) \varphi(t) + \sum_{j=1}^n |a_{ij}(t)| |F_j| \varphi(t) + \sum_{j=1}^n |b_{ij}(t)| |G_j| \frac{V(q_{ij} t)}{(q_{ij} t + r)^p} \\
&\le -\left[d_i(t) - \sum_{j=1}^n |a_{ij}(t)| |F_j| \right] \varphi(t) + \sum_{j=1}^n |b_{ij}(t)| |G_j| \frac{V(q_{ij} t)}{(q_{ij} t + r)^p}.
\end{aligned}
$$

For $V(t)$

$$
\begin{aligned}
{}_0^C D_t^p V(t) &\le (t+r)^p \, {}_0^C D_t^p \varphi(t) + \frac{(p^2 + p^{-1}) \hat{V}(t)}{r^p \Gamma(2-p)} \\
&\le (t+r)^p \left\{ -\left[d_i(t) - \sum_{j=1}^n |a_{ij}(t)| |F_j| \right] \varphi(t) \right. \\
&\quad \left. + \sum_{j=1}^n |b_{ij}(t)| |G_j| \frac{V(q_{ij} t)}{(q_{ij} t + r)^p} \right\} + \frac{(p^2 + p^{-1}) \hat{V}(t)}{r^p \Gamma(2-p)} \qquad (12) \\
&\le \left\{ \begin{aligned} & -d_i(t) + \sum_{j=1}^n \left(|a_{ij}(t)| |F_j| + |b_{ij}(t)| \frac{|G_j|}{q_{ij}^p} \right) \\ & + \frac{(p^2 + p^{-1})}{r^p \Gamma(2-p)} \end{aligned} \right\} \hat{V}(t) \\
&\le \omega \hat{V}(t).
\end{aligned}
$$

where

$$
\omega = -d_i(t) + \sum_{j=1}^n \left(|a_{ij}(t)| |F_j| + |b_{ij}(t)| \frac{|G_j|}{q_{ij}^p} \right) + \frac{(p^2 + p^{-1})}{r^p \Gamma(2-p)}.
$$

If $V(t) = V(t)$, estimation(12) gives ${}_0^C D_t^p V(t) \le 0$. If $V(t) < \hat{V}(t)$, then there exists a $\delta > 0$, such that $V(t+s) < \hat{V}(t), \forall s \in [0, \delta)$. Therefore, $V(t+\theta) \le V(t)$ for all $\theta \in [0, \delta)$, which also yields ${}_0^C D_t^p V(t) \le 0$. We can conclude by the above argument that ${}_0^C D_t^p V(t) \le 0$. holds for all $t > 0$, and hence $V(t) \le V(0)$.

So

$$
\begin{aligned}
|y_i(t) - x_i(t)| &= \frac{V(t)}{(t+r)^p} \le \frac{V(t)}{(t+r)^p} \le \frac{V(0)}{(t+r)^p} \\
&= \frac{r^P \varphi(0)}{(t+r)^p} \le \frac{r^P |x_1^0 - x_2^0|}{(t+r)^p}, \quad \forall t \ge 0.
\end{aligned}
$$

Hence,

$$
\|y_i(t) - x_i(t)\| \le \left\| \frac{r^p}{(t+r)^p} \right\| \|x_1^0 - x_2^0\|
$$

when $\|x_1^0 - x_2^0\| < \delta$ and $\left\| \frac{r^p}{(t+r)^p} \right\| < \frac{\varepsilon}{\delta}$, $\|y_i(t) - x_i(t)\| < \varepsilon.$

According to **Definition 3.1**, the system (1) is finite-time stable.

4. Numerical Examples

In this section, we give two numerical examples to verify the validity of the results.

Example 1. Consider the following non-autonomous fractional neural network model

$$
\begin{aligned}
{}_0^C D_t^{0.9} x_i(t) &= -d_i x_i(t) + \sum_{j=1}^2 a_{ij} \tanh(x_j(t)) \\
&\quad + \sum_{j=1}^2 b_{ij} \tanh(x_j(0.8t)) + I_i(t), \quad t > 0
\end{aligned}
$$

The system parameters d_i, a_{ij}, b_{ij} are given by

$$D=\operatorname{diag}(0.5,0.5),\ A=\begin{pmatrix}1 & -0.2\\ -0.4 & 0.8\end{pmatrix},$$

$$B=\begin{pmatrix}0.1 & -0.5\\ 0.3 & 0.1\end{pmatrix},\ I=(-10,5)^T,$$

the neuron activation functions $f_i(x)=g_i(x)=\tanh(x_i), i=1,2.$ where $f_j=g_j=1,\ p=0.9.$ Thus

$$-K=D-(|A|+q^{-p}|B|)=\begin{pmatrix}1.8778 & -0.8112\\ -0.7667 & 3.0778\end{pmatrix}.$$

K is a nonsingular M - matrix. Next it needs to verity the finite-time stability w.r.t.

$\{t_0=0;\delta=0.1;\varepsilon=1;q=0.8\}$, where $\dfrac{r^p}{(t+r)^p}<\dfrac{\varepsilon}{\delta}$, the system satisfies all the assumptions, Simulation results with six groups initial conditions of $x_1(t),x_2(t)$

$$\begin{Bmatrix}(20,9)^T,(14,-14)^T,(9,-4.5)^T,\\ (-9,18.7)^T,(-14.5,13.9)^T,(-5,10)^T\end{Bmatrix}.$$

Which show that the neural network system with the above parameters is finite time stable, see Fig 1. and Fig 2.

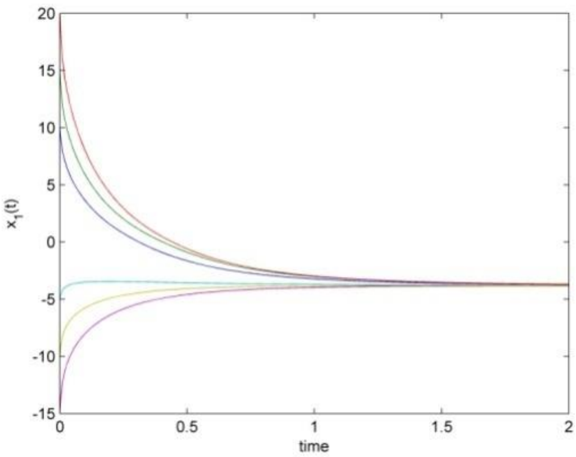

Fig. 1. Trajectories of $x_1(t)$ with $p=0.9$

Example 2 Consider the following model of FONNS

$$ {}_0^C D_t^{0.9} x_i(t)=-d_i x_i(t)+\sum_{j=1}^{3} a_{ij}\tanh(x_j(t))$$

$$+\sum_{j=1}^{3} b_{ij}\tanh(x_j(0.8t))+I_i(t),\quad t>0$$

where $p=0.9.$ The system parameters d_i, a_{ij}, b_{ij} are given by

$$D=\operatorname{diag}(4,3.5,3.8),\ A=\begin{pmatrix}0.8 & -0.4 & -1\\ -0.1 & -0.1 & 1.1\\ 0.4 & -0.8 & 0.4\end{pmatrix},$$

$$B=\begin{pmatrix}0.1 & -0.25 & -0.5\\ 0.15 & 0.1 & 0.1\\ 0.2 & -0.4 & -0.15\end{pmatrix},\ I=(10,-10,10)^T,$$

the neuron activation functions $f_i(x)=g_i(x)=\tanh(x_i)$, $i=1,2,3.$ Note that $f_j=g_j=1,$ thus

$$-K=D-(|A|+q^{-p}|B|)=\begin{pmatrix}3.0778 & -0.7056 & -1.6112\\ -0.2834 & 3.2778 & -1.2222\\ -0.6445 & -1.2890 & 3.2166\end{pmatrix}.$$

K is a nonsingular M -matrix.

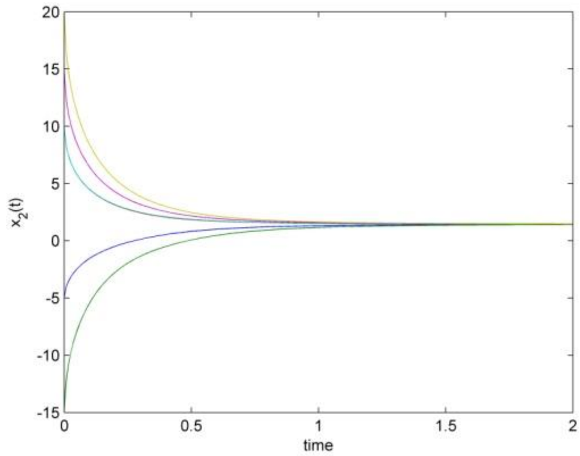

Fig. 2. Trajectories of $x_2(t)$ with $p=0.9$

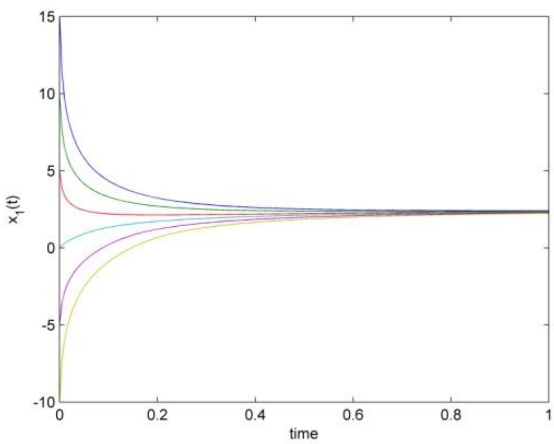

Fig. 3. Trajectories of $x_1(t)$ with $q=0.8$

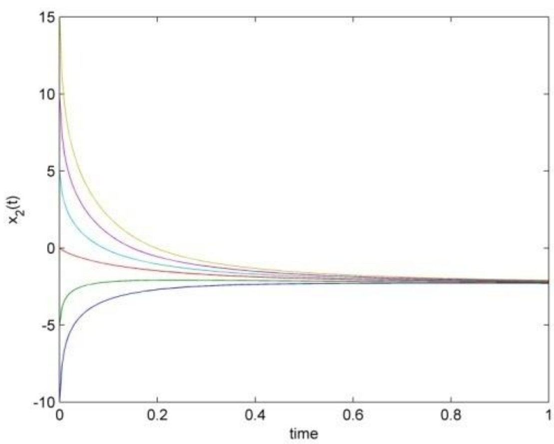

Fig. 4. Trajectories of $x_2(t)$ with $q=0.8$

Note that $\{t_0=0;\delta=0.1;\varepsilon=1;q=0.8\}$, where $\dfrac{r^p}{(t+r)^p}<\dfrac{\varepsilon}{\delta}.$

In finite-time, the system satisfies all the assumptions. Simulation results with six groups initial conditions of $x_1(t),x_2(t),x_3(t)$,

$$\begin{Bmatrix} (5,0,3)^T, (7.3,-3.5,3.5)^T, (11.2,-7.5,7.5)^T, \\ (0,3.4,-3.2)^T, (6.3,7.2,-7.5)^T, (-7.4,11,-11.5)^T \end{Bmatrix}.$$

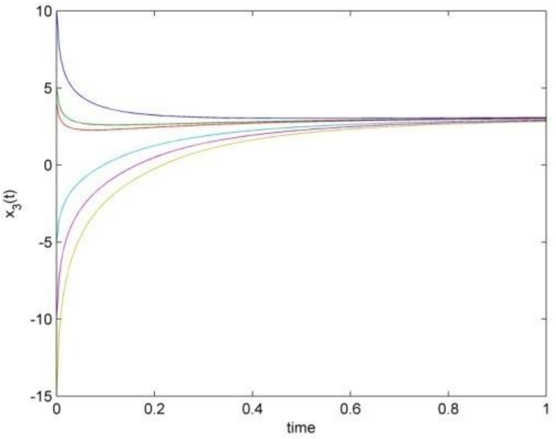

Fig. 5. Trajectories of $x_3(t)$ with $q = 0.8$

Which show that the neural network system with the above parameters is finite time stable, see Fig 3-5.

5. Conclusions

In this paper, we study the finite-time stability of fractional non-autonomous neural networks with heterogeneous proportional delays. According to the homeomorphism mapping principle, the nonlinear mapping is constructed, and the existence of unique solution is proved. By using the extended comparison technique of Leibniz rule for fractional differential, a time-series neural network model based on delay is derived to ensure the global stability of the considered fractional neural network model. Two numerical examples are given to verify the effectiveness of the results.

Acknowledgements

I would like to express my gratitude to all those who have helped me during the writing of this paper.

References

Podlubny, I., 1999. Fractional Differential Equations. Academic, New York.

Aghababa, M., Khanmohammadi, S., Alizadeh, G., 2011. Finite-time synchronization of two different chaotic systems with unknown parameters via sliding mode technique. Applied Mathematical Modeling. 35, 3080-3091.

Arena, P., Fortuna, L., Porto, D., 2000. Chaotic behavior in nonintegerorder cellular neural networks. Phys. Rev. E. 61, 777-781.

Boroomand, A., Menhaj, M., 2009. Fractional-order Hopfield neural networks. Lect. Notes. Sci. 5506, 883-890.

Cao, J., Zhou, D.,1998. Stability analysis of delayed cellular neural networks. Neural networks. 11, 1601-1605.

Corduneanu, C., 1971. Principles of Different and Integral Equations. Allyn and Bacon. Boston.

Delavari, H., Balcanu, D., Sadati, S. J., 2012. Stability analysis of Caputo fractional-order nonliner systems revisited. Nonliner Dynam. 67, 2433-2439.

Diethelm, K., 2010. The Analysis of Fractional Differential Equations: An Application-oriented Exposition using Differential Operators of Caputo type. Springer. Berlin. Heidelberg.

Gupta, M., Jin, L., Homma, N., 2003. Static and Dynamic Neural Networks. Wiley-Interscience. New York.

Hien, L. V., Son, D. T., 2015. Finite-time stability of a class of non-autonomous neural networks with heterogeneous proportional delays. Appl Math Comput. 251, 14-23.

Koeller, R. C., 1984. Applications of fractional calculus to the theory of viscoelasticity. J. Appl. Mech. 51, 299-307.

Ma, Y., Chen, H., 2015. Reliable finite-time H∞ filtering for discrete time-delay systems with Markovian jump and randomly occurring nonlinearities. Applied Mathematics and Computation. 268, 897-915.

Li, C., Deng, W.,2007. Remarks on fractional derivatives. Applied Mathematics Computation. 187, 777-784.

Li, Y., Chen, Y. Q., Podlubny, I., 2009. Mittag-Leffler stability of fractional order nonlinear dynamic system. Automatica. 45, 1965-1969.

Liu, B., 2017. Finite-time stability of a class of CNNs with heterogeneous proportional delays and oscillating leakage coefficients. Neural Process Lett. 45, 109-119.

Luo, J. W., 2007. Fixed points and stability of neutral stochastic delay differential equations. J. Math. Anal. Appl. 334, 431-440.

LUO, X., GE, S., WANG, J. & GUAN 2018. Time Delay Estimation-based Adaptive Sliding-Mode Control for Nonholonomic Mobile Robots. International Journal of Applied Mathematics in Control Engineering, 1, 1-8.

Mandelbort, B. B., 1982. The Fractal Geometry of Nature. W.H. Freeman and Company. New York.

Matignon, D., 1996. Stability Restults for Fractional Differential Equations with Applications to Control Processing//Computational Engineering in Systems Applications. Lille. France. IMACS. IEEE-SMC. 2, 963-968.

Petras, I., 2011. Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation. Springer. Berlin. Heidelberg.

Podlubny, I., 1999. Fractional Differential Equations. Academic, New York.

Ren, D. B., Zhang, J. Y., 2007. Global exponential stability of a class of neural networks with distributed delays.Computer Science. 34, 159-161.

Samko, S. G., Kilbas, A. A., Marichev, Q. I.,1993. Fractional integrals and derivatives. Theory and Applications. Gordon and Breach. Yverdon. 1993, 44.

Shi, M., Wang, Z. H., 2011. An effective analytical criterion for stability testing of fractional delay systems. Automatica. 47, 2000-2006.

SHI, S., LIU, M. & MA, J. 2018. Fractional-Order Sliding Mode Control for a Class of Nonlinear Systems via Nonlinear Disturbance Observer. International Journal of Applied Mathematics in Control Engineering, 1, 103-110.

Wang, H., Yu, Y., Wen, G., et al., 2015. Global stability analysis of fractional-order Hopfield neural networks with time delay. Neurocomputing. 154, 15-23.

Wu, R.C., Hei, X.D., Chen, L.P., 2013. Finite-time stability of fractional-order neural networks with delay. Commun. Theor. Phys. 60, 189-193.

WU, W. J. 2018. Parametric Lyapunov Equation Approach to Stabilization and Set Invariance Conditions of Linear Systems with Input Saturation. International Journal of Applied Mathematics in Control Engineering, 1, 79-84.

Yan, L., Chen, Y. Q., Podlubny, I., 2010. Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability. Computers and Mathematics with Applications. 59, 1805-1820.

Yang, Z., Cao, J., 2013. Initial value problems for arbitrary order fractional equations with delay. Commun Nonlinear Sci Numer Simulat. 18:2993–3005.

Yu, J., Hu, C., Jiang, H. J., et al., 2014. Projective synchronization for fractional neural networks. Neural Networks. 49, 87-95.

Yu, J., Hu, H., Zhou, S., et al., 2013. Generalized Mittag-Leffler stability of Multi-Variables fractional order nonlinear systems. Automatica. 49,1798-1803.

Wen Shengnan is currently pursuing her MS study at the College of Science, Yanshan University, Qinhuangdao, China. She obtained her B.S. degree from Yanshan University, Qinhuangdao, in 2015. Her main research interests are in the areas of stability of fractional neural networks.

Liu Deyou received the B.S. degree in Mathematics from Harbin Institute of Technology in 1986, and the Ph.D. degree in control science and engineering from Yanshan University in 2008. He is currently a professor with the School of Science, Yanshan University. His main research interests are in the areas of control theory and application.

Wang Xiaoyuan is currently pursuing his MS study at School of Scicence, Yanshan University, Qinhuangdao, China. He obtained his B.S. degree in Mathematics and Applied Mathematics from Shijiazhuang University in 2015. His main research interests are in the areas of Complex analysis and Application.