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Phase Trajectory Analysis of Sliding Mode Control with Unidirectional Auxiliary Surfaces Method

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ABSTRACT

A sliding mode control with unidirectional auxiliary surfaces (UAS-SMC) method is applied to the second-order nonlinear system. The method of partition discussion is used to study the operation law of the phase trajectory, and the concept and explanation expression of the partition surface are given. Based on the distribution of phase trajectories, the influence of the initial point and k-value of the approaching law on the phase trajectory is analyzed, and the motion trend of phase trajectories passing through two switching surfaces is discussed. simulations are given to show Correctness of analysis results.

1. Introduction

Sliding mode control is a robust nonlinear control method, which guarantees the dynamic performance of the system by designing the sliding mode dynamics of the system. Because of its insensitivity to parameter uncertainty and external disturbance, good robustness, simple physical implementation and fast response, it has been widely used in various nonlinear system control (e.g., Yang et al., 2012; Utkin, V., 1977). Such as spacecraft (e.g., Cheema et al., 2017), PMSM speed controller (e.g., Pukdeboon et al., 2009), magnetic bearing system (e.g., Tapan et al., 2018), robot (e.g., Luo, X., et al, 2018), Quadrotor UAV (e.g., Chen, Q., et al, 2018) and so on.

Sliding mode control with unidirectional auxiliary surfaces is a polyhedron positive invariant set of system state constructed with unidirectional auxiliary surfaces, so as to ensure that the system state and control input can meet the constraints in the whole process (e.g., Fu et al., 2011). This method can avoid chattering on the switching surface under certain conditions, but it will slow down the convergence speed to a certain extent. In recent years, UAS-SMC has made some progress. Exponential approaching law and double power approaching law is used to speed up convergence (Hu et al., 2013; Zhang et al., 2018); Introduced the state constraints into the unidirectional auxiliary surfaces, so that the system state can meet the constraints in the whole operation process (e.g., Fu J et al., 2011). Design of controller based on sliding mode control with unidirectional auxiliary surfaces and nonlinear disturbance observer, to reduce chattering and improve convergence speed significantly (e.g., Ren et al., 2013). Sliding mode control with unidirectional auxiliary surfaces is mainly used in aircraft control (e.g., He et al., 2015), synchronous motor control (e.g., Liu et al., 2018.), hypersonic vehicle (e.g., Yang, Z., et al., 2018). Although UAS-SMC has been applied, its understanding is not enough. Because it has two switching surfaces and four unidirectional auxiliary surfaces, its phase trajectory may be more complex, and the purpose of the two switching surfaces is unknown. This paper mainly studies the phase trajectory of sliding mode control with unidirectional auxiliary surfaces method.

In this paper, a sliding mode controller with unidirectional auxiliary surfaces based on constant approaching law is designed for the second-order nonlinear system. At the same time, the concept and explanation expression of the segmentation surface in the phase plane are described. Then, by the method of partition analysis, the distribution of phase trajectories under different initial points and k-values of approaching law is analyzed. And the motion trend of phase trajectory when passing through two sliding surfaces is also analyzed. Finally, simulations are given to show correctness of analysis results.

2. Numerical approach

2.1 Description of System

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Consider a class of non-linear system as follow:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(\boldsymbol{x}) + g(\boldsymbol{x}) \cdot u \end{cases} \tag{1}$$

Where $\boldsymbol{x} \in [x_1, x_2]^T$ indicates the state of the system. $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ are nonlinear function of state $\boldsymbol{x}$, and $|g(\boldsymbol{x})| \neq 0$ .While u is the control inputs.

2.2 Design Steps of UAS-SMC System

The whole process is divided into three steps:

Step 1

The switching surfaces for the state $\boldsymbol{x}$ in system (1) are given by

$$\begin{cases} S_1 = x_2 + \xi_1 x_1 = 0 \\ S_2 = x_2 + \xi_2 x_1 = 0 \end{cases} \tag{2}$$

Where $\xi_2 > \xi_1 > 0$,and it is used to avoid the overlap of switching surfaces $S_1(\boldsymbol{x}) = 0$ and $S_2(\boldsymbol{x}) = 0$. Based on the switching surface S_1, S_2 ,the $No.0_i, ..., 3_i$ subspaces can be defined in figure 1. The four subspace are defined by $Eq.(3)$:

$$\begin{cases} No.0 = \{(x_2, x_1) | S_1 < 0, S_2 < 0\} \\ No.1 = \{(x_2, x_1) | S_1 \geq 0, S_2 < 0\} \\ No.2 = \{(x_2, x_1) | S_1 < 0, S_2 \geq 0\} \\ No.3 = \{(x_2, x_1) | S_1 \geq 0, S_2 \geq 0\} \end{cases} \tag{3}$$

Step 2

Select four appropriate points P_1, P_2, P_3, P_4 on switching surfaces S_1, S_2 ,where point P_1, P_2 are located in the fourth quadrant and points P_3, P_4 are located in the second quadrant , as shown in Figure 1. there exists $Eq.(4)$

$$\begin{aligned} S_1(P_1) = S_1(P_3) = 0 \\ S_2(P_2) = S_2(P_4) = 0 \end{aligned} \tag{4}$$

The line P_1P_4 , P_1P_2 , P_3P_4 , P_2P_3 defined in Figure 2 are UAS h_0, h_1, h_2, h_3 . The formulas for these UAS are given as follows:

$$h_i = \omega_{i1} x_2 + \omega_{i2} x_1 + w \quad i = 0,1,2,3 \tag{5}$$

Where

$$\omega_{i1} = \begin{cases} \omega_{01} & S_1 < 0, S_2 < 0 \\ \omega_{11} & S_1 \geq 0, S_2 < 0 \\ \omega_{21} & S_1 < 0, S_2 \geq 0 \\ \omega_{31} & S_1 \geq 0, S_2 \geq 0 \end{cases}$$

$$\omega_{i2} = \begin{cases} \omega_{02} & S_1 < 0, S_2 < 0 \\ \omega_{12} & S_1 \geq 0, S_2 < 0 \\ \omega_{22} & S_1 < 0, S_2 \geq 0 \\ \omega_{32} & S_1 \geq 0, S_2 \geq 0 \end{cases}$$

ω_{i1}, ω_{i2} are the coefficients of h_i , and w is a constant greater than zero. i denotes the number of subspaces. The coefficients ω_{11}, ω_{21} in $Eq.(5)$ should satisfy $\omega_{11} < 0$, $\omega_{21} > 0$, which is a sufficient condition for the existence of chattering-free control input (i.e., Fu, et al, 2013).

Step 3.

The UAS-SMC control input u for the nonlinear system (1) is designed by solving the equation:

$$\begin{aligned} \dot{h}_i &= \omega_{i1} \dot{x}_2 + \omega_{i2} \dot{x}_1 \\ &= \omega_{i1} \left(f(\boldsymbol{x}) + g(\boldsymbol{x}) \cdot u \right) + \omega_{i2} x_2 = N \end{aligned} \tag{6}$$

Where N is the approaching law ,and it is designed as follow:

$$N = k(w - h_i) \quad i = 0,1,2,3 \quad k > 0 \tag{7}$$

It follows that the UAS-SMC control input for system (1) is expressed as:

$$u = (g(\boldsymbol{x}))^{-1} \left(-f(\boldsymbol{x}) + \omega_{i1}^{-1} (-\omega_{i2} x_2 + N) \right) \tag{8}$$

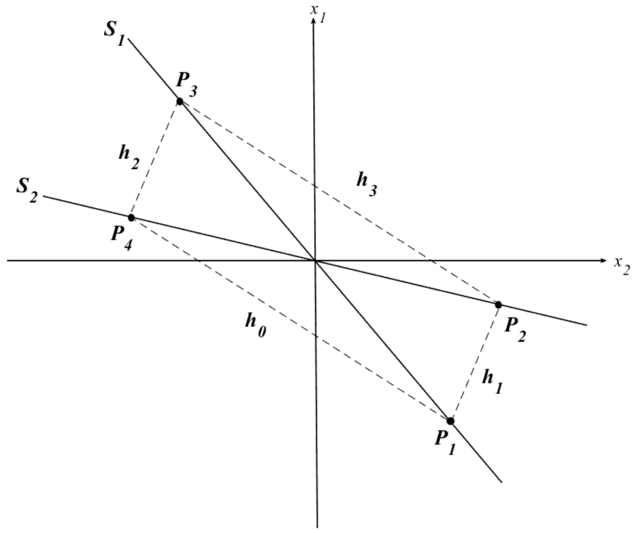

Fig.1. SMC with UASs

3. Trajectory analysis of UAS-SMC system

For system (1), substituting $Eq.(5)$, $Eq.(7)$ and $Eq.(8)$ into $Eq.(1)$, leads to

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \omega_{i1}^{-1} (-\omega_{i2} x_2 + N) \\ \quad = \omega_{i1}^{-1} \left(-\omega_{i2} x_2 + k(-\omega_{i1} x_2 - \omega_{i2} x_1) \right) \\ \quad = -k x_2 - \omega_{i1}^{-1} \omega_{i2} (x_2 + k x_1) \end{cases} \tag{9}$$

Define $\dot{x}_2 = -k x_2 - \omega_{i1}^{-1} \omega_{i2} (x_2 + k x_1) = m_i$, where $i = 0,1,2,3$ is the number of subspaces.

From $Eq.(9)$ the main influencing factors of the system is initial point and the value of k . For the convenience of phase trajectory analysis, based on the coordinate axis, the $No.0_i, No.3_i$ subspaces can be defined in figure 2.

Where

$$\begin{cases} No.0_{ia} = \{(x_2, x_1) | S_1 < 0, S_2 < 0, x_2 < 0, x_1 > 0\} \\ No.0_{ib} = \{(x_2, x_1) | S_1 < 0, S_2 < 0, x_2 \leq 0, x_1 \leq 0\} \\ No.0_{ic} = \{(x_2, x_1) | S_1 < 0, S_2 < 0, x_2 > 0, x_1 < 0\} \end{cases} \tag{10}$$

$$\begin{cases} No.3_{ia}=\{(x_2,x_1)|S_1>0,S_2>0,x_2>0,x_1<0\} \\ No.3_{ib}=\{(x_2,x_1)|S_1>0,S_2>0,x_2\ge 0,x_1\ge 0\} \\ No.3_{ic}=\{(x_2,x_1)|S_1>0,S_2>0,x_2<0,x_1>0\} \end{cases} \tag{11}$$

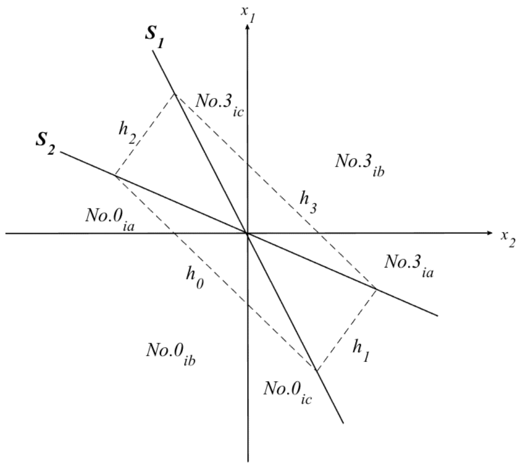

Fig.2. UASs after repartition

In order to study the influence of different initial point on the trajectory conveniently, the line between the origin and the initial point is used as guide line, which can be defined as Partition surface:

$$x_2(0)+ax_1(0)=0 \tag{12}$$

Where $(x_2(0),x_1(0))$ is the initial point of the trajectory.

If the trajectory is close to the switching surface, the expression of definition $\dot{x}_2$ is $\dot{x}_{2\_No.i_j}$. Where i,j is the number of subspaces.

Theorem 1. If the k value of approaching law satisfies the condition $k=a$,the trajectory will reach the origin along the partition surface.

Prove. If $k=a$, there are $x_2(0)+kx_1(0)=0$ and $Eq.(13)$.

$$\begin{cases} \dot{x}_1=x_2 \\ \dot{x}_2=-kx_2-\omega_{i1}^{-1}\omega_{i2}(x_2+kx_1) \end{cases} \tag{13}$$

Simplified $Eq.(13)$, the $Eq.(14)$ can be obtained:

$$\ddot{x}_1+(k+\omega_{i1}^{-1}\omega_{i2})\dot{x}_1+k\omega_{i1}^{-1}\omega_{i2}x_1=0 \tag{14}$$

Solved $Eq.(14)$,There are two general solutions of x_1 and x_2 :
If $k\ne\omega_{i1}^{-1}\omega_{i2}$, the $Eq.(15)$ can be obtained:

$$\begin{cases} x_1=C_1e^{-kt}+C_2e^{-\omega_{i1}^{-1}\omega_{i2}t} \\ x_2=-kC_1e^{-kt}-\omega_{i1}^{-1}\omega_{i2}C_2e^{-\omega_{i1}^{-1}\omega_{i2}t} \end{cases} \tag{15}$$

Substituted $t=0$ into $Eq.(15)$, the $Eq.(16)$ can be obtained:

$$\begin{cases} x_1(0)=C_1+C_2 \\ x_2(0)=-kC_1-\omega_{i1}^{-1}\omega_{i2}C_2 \end{cases} \tag{16}$$

Solved $Eq.(16)$ and $x_2(0)+kx_1(0)=0$, can get $C_1=x_1(0)=-x_2(0)/k$, $C_2=0$, and the solution of x_1 and x_2 is shown in $Eq.(17)$

$$\begin{cases} x_1=-\dfrac{x_2(0)e^{-kt}}{k}=-x_1(0)e^{-kt} \\ x_2=x_2(0)e^{-kt} \end{cases} \tag{17}$$

If $k=\omega_{i1}^{-1}\omega_{i2}$, the $Eq.(18)$ can be obtained:

$$\begin{cases} x_1=(C_1+C_2t)e^{-kt} \\ x_2=-kC_1e^{-kt}-kC_2te^{-kt}+C_2e^{-kt} \end{cases} \tag{18}$$

Substituted $t=0$ into $Eq.(18)$, the $Eq.(19)$ can be obtained:

$$\begin{cases} x_1(0)=C_1+C_2 \\ x_2(0)=-kC_1+C_2 \\ x_2(0)+kx_1(0)=0 \end{cases} \tag{19}$$

Solved $Eq.(18)$ and $Eq.(19)$, the solution of x_1 and x_2 are the same as those of $Eq.(17)$.

From $Eq.(17)$, the trajectory run law of x_2 and x_1 always satisfies $x_2+kx_1=x_2+ax_1=0$. Therefore, the trajectory reaches the origin follow the partition surface $x_2+ax_1=0$. Because of $k>0$, this theorem is only possible in the second and fourth quadrants.

Next, the influence of initial point and k value of four subspace on phase trajectory is analyzed.

3.1 trajectory analysis of $No.0_i$ *subspace*

When the initial point is in $No.0_i$ subspace, from the definition of UAS, can get $\omega_{01}>0,\omega_{02}>0$, $\xi_2>\omega_{01}^{-1}\omega_{02}>\xi_1$ and there are three parts in the subspace:

(1).When the initial point is in $No.0_{ia}$ subspace, can get $\xi_2>\xi_1>a$. There are three situations:

Combined with known conditions, and substituting $Eq.(10)$ into $Eq.(9)$, leads to

$$\begin{cases} \dot{x}_1=x_2<0 \\ \dot{x}_2=\omega_{01}^{-1}(-\omega_{02}x_2+N)>0 \end{cases} \tag{20}$$

Due to $\dot{x}_1<0,\dot{x}_2>0$, initial point moves along the negative direction of x_1 axis and the positive direction of x_2 axis. With the increase of k value, there are three situations:

①.The value of k is in interval $(0,a)$, can get $x_2+kx_1<0$. Partial derivative of m_0 to k , the $Eq.(21)$ can be obtained:

$$\frac{\partial m_0}{\partial k}=-(x_2+\omega_{01}^{-1}\omega_{02}x_1) \tag{21}$$

Combined with known conditions, can get $x_2+\omega_{01}^{-1}\omega_{02}x_1<0$,and $\partial m_0/\partial k>0$, m_0 is an increasing function. The maximum value is:

$$\lim_{k\to a}m_0=\lim_{k\to a}\left(-kx_2-\omega_{i1}^{-1}\omega_{i2}(x_2+kx_1)\right)=-ax_2 \tag{22}$$

Where $0<k<a$,can get $\dot{x}_2<-ax_2=-a\dot{x}_1$ .The increasing speed of x_2 is less than a times of the decreasing speed of x_1 . At this time, the trajectory is always below partition surface $x_2+ax_1=0$.

When x_1 is zero, x_2 is still less than zero, at this time, the point will cross the negative axis of x_2 to the $No.0_{ib}$ subspace.

As the trajectory moves to $No.0_{ib}$ subspace, combined with known conditions, can get $\dot{x}_1 < 0, \dot{x}_2 > 0$, the trajectory moves along the negative direction of x_1 axis and the positive direction of x_2 axis. When x_2 is zero, x_1 is still less than zero. Therefore, the point will cross the negative axis of x_1 to the $No.0_{ic}$ subspace.

As the trajectory moves to $No.0_{ic}$ subspace, because of $\dot{x}_1 = x_2 > 0$, x_1 will increase and move along the positive direction of x_1 axis. When x_2 is close to zero, the $Eq.(23)$ can be obtained:

$$\lim_{x_2 \to 0} m_0 = \lim_{x_2 \to 0}\left(-kx_2 - \omega_{01}^{-1}\omega_{02}\left(x_2 + kx_1\right)\right) = -\omega_{01}^{-1}\omega_{02}kx_1 \quad (23)$$

Where $\omega_{01} > 0, \omega_{02} > 0, k > 0, x_1 < 0$, can get $\dot{x}_2 > 0$. x_2 will increase and move along the positive direction of x_2 axis. There are two situations:

If the value of k is in interval $(0, \xi_1]$, with the increase of x_1, there must be a point (x_2, x_1) that satisfies $x_2 + kx_1 = 0$. At this time, there are $\dot{x}_2 = -kx_2 = -k\dot{x}_1$ and $x_2 + \mathrm{k}x_1 = 0$. Similar to theorem 1, the trajectory reaches the origin along line $x_2 + \mathrm{k}x_1 = 0$.

If the value of k is in interval (ξ_1, a), with the increase of x_1, the trajectory must reach the switching surface S_1. When the trajectory reaches the switching surface S_1, there are $No.1_i$ subspace and $No.0_{ic}$ subspace above and below the switching surface. Define $\dot{x}_{2\text{-}No.0_{ic}} = \mathrm{m}_a$, $\dot{x}_{2\text{-}No.1_i} = \mathrm{m}_b$. Then the $Eq.(24)$ can be obtained:

$$\begin{cases} \mathrm{m}_a = -kx_2 - \omega_{01}^{-1}\omega_{02}\left(x_2 - \dfrac{kx_2}{\xi_1}\right) \\ \mathrm{m}_b = -kx_2 - \omega_{11}^{-1}\omega_{12}\left(x_2 - \dfrac{kx_2}{\xi_1}\right) \end{cases} \quad (24)$$

Partial derivative of m_a and m_b to k, the $Eq.(25)$ can be obtained:

$$\begin{cases} \dfrac{\partial m_a}{\partial k} = \left(-1 + \dfrac{\omega_{01}^{-1}\omega_{02}}{\xi_1}\right)x_2 \\ \dfrac{\partial m_b}{\partial k} = \left(-1 + \dfrac{\omega_{11}^{-1}\omega_{12}}{\xi_1}\right)x_2 \end{cases} \quad (25)$$

Combined with known conditions, can get $\partial m_a / \partial k > 0$, m_a is an increasing function. The minimum value is:

$$\lim_{k \to \xi_1} m_a = \lim_{k \to \xi_1}\left(-kx_2 - \omega_{01}^{-1}\omega_{02}\left(x_2 - \frac{kx_2}{\xi_1}\right)\right) = -\xi_1 x_2 \quad (26)$$

Where $k < a$, it is easy to get $\dot{x}_{2\text{-}No.0_{ic}} > -\xi_1 x_2$, $\dot{x}_{2\text{-}No.0_{ic}} / \dot{x}_{1\text{-}No.0_{ic}} > -\xi_1$.

When the trajectory moves to $No.1_i$ subspace, can get $\omega_{11} < 0$. If $\omega_{12} > 0$, it is easy to get $\partial m_b / \partial k < 0$; If $\omega_{12} < 0$, it is easy to get $\partial m_b / \partial k < 0$. m_b is an decreasing function. The maximum value is：

$$\lim_{k \to \xi_1} m_b = \lim_{k \to \xi_1}\left(-kx_2 - \omega_{01}^{-1}\omega_{02}\left(x_2 - \frac{kx_2}{\xi_1}\right)\right) = -\xi_1 x_2 \quad (27)$$

Where $k < a$, it is easy to get $\dot{x}_{2\text{-}No.1_i} < -\xi_1 x_2$, $\dot{x}_{2\text{-}No.1_i} / \dot{x}_{1\text{-}No.1_i} < -\xi_1$.

Above and below the switching surface S_1 ,there are $\dot{x}_{2\text{-}No.1_i} / \dot{x}_{1\text{-}No.1_i} < -\xi_1$ and $\dot{x}_{2\text{-}No.0_{ic}} / \dot{x}_{1\text{-}No.0_{ic}} > -\xi_1$ .Therefore, when the trajectory reaches the switching surface S_1, it will follow the switching surface S_1 to the origin.

②.When $k = a$, it conforms to theorem 1. The trajectory reaches the origin along the partition surface $x_2 + ax_1 = 0$.

③.The value of k is in interval $(\mathrm{a}, +\infty)$, can get $x_2 + kx_1 > 0$. Combined with known conditions, it is easy to get $\dot{x}_2 > -a\dot{x}_1$. The increasing speed of x_2 is more than a times of the decreasing speed of x_1. At this time, the trajectory is always above the partition surface $x_2 + ax_1 = 0$, and must reach the switching surface S_2.

As the trajectory reaches the switching surface S_2, there are $No.2_i$ subspace and $No.0_{ia}$ subspace above and below the switching surface. When the trajectory moves to the switching surface S_2, the $Eq.(28)$ can be obtained:

$$\begin{cases} \dot{x}_{2\text{-}No.0_{ia}} = -kx_2 - \Omega_{01}^{-1}\Omega_{02}\left(x_2 + kx_1\right) \\ \dot{x}_{2\text{-}No.2_i} = -kx_2 - \Omega_{21}^{-1}\Omega_{22}\left(x_2 + kx_1\right) \end{cases} \quad (28)$$

Combined $\Omega_{01}^{-1}\Omega_{02} > \Omega_{11}^{-1}\Omega_{12}$, $\Omega_{11}^{-1}\Omega_{12} = \Omega_{21}^{-1}\Omega_{22}$, $x_2 + kx_1 > 0$ and $Eq.(28)$, can get $\dot{x}_{2\text{-}No.0_{ia}} < \dot{x}_{2\text{-}No.2_i}$. Therefore, the trajectory will become more gentle, and it will surely pass through the switching surface S_2 to the switching surface S_1.

When the trajectory reaches the switching surface S_1, there are $No.3_{ic}$ subspace and $No.2_i$ subspace above and below the switching surface. Define $\dot{x}_{2\text{-}No.3_{ic}} = \mathrm{m}_c$, $\dot{x}_{2\text{-}No.2_i} = \mathrm{m}_d$. Then the $Eq.(29)$ can be obtained:

$$\begin{cases} \mathrm{m}_c = -kx_2 - \Omega_{31}^{-1}\Omega_{32}\left(x_2 - \dfrac{kx_2}{\xi_1}\right) \\ \mathrm{m}_d = -kx_2 - \Omega_{21}^{-1}\Omega_{22}\left(x_2 - \dfrac{kx_2}{\xi_1}\right) \end{cases} \quad (29)$$

Partial derivative of m_c and m_d to k, the $Eq.(30)$ can be obtained:

$$\begin{cases} \dfrac{\partial m_c}{\partial k} = \left(-1 + \dfrac{\Omega_{31}^{-1}\Omega_{32}}{\xi_1}\right)x_2 \\ \dfrac{\partial m_d}{\partial k} = \left(-1 + \dfrac{\Omega_{21}^{-1}\Omega_{22}}{\xi_1}\right)x_2 \end{cases} \quad (30)$$

Combined with known conditions, it is easy to get $\partial m_c / \partial k < 0$, m_c is an decreasing function. The maximum value is:

$$\lim_{k \to \xi_1} m_c = \lim_{k \to \xi_1}\left(-kx_2 - \omega_{01}^{-1}\omega_{02}\left(x_2 - \frac{kx_2}{\xi_1}\right)\right) = -\xi_1 x_2 \quad (31)$$

Where $k > a > \xi_1$, can get $\dot{x}_{2\text{-}No.3_{ic}} < -\xi_1 x_2$, $\dot{x}_{2\text{-}No.3_{ic}} / \dot{x}_{1\text{-}No.3_{ic}} > -\xi_1$.

When trajectory moves to $No.2_i$ subspace, can get $\omega_{21} > 0$. If $\omega_{22} < 0$, it is easy to get $\partial m_d / \partial k > 0$; If $\omega_{22} > 0$, it is easy to get $\partial m_d / \partial k > 0$. m_d is an increasing function. The minimum value is

$$\lim_{k \to \xi_1} m_b = \lim_{k \to \xi_1}\left(-kx_2 - \omega_{01}^{-1}\omega_{02}\left(x_2 - \frac{kx_2}{\xi_1}\right)\right) = -\xi_1 x_2 \quad (32)$$

Where $k > a > \xi_1$, can get $\dot{x}_{2\text{-}No.1_i} > -\xi_1 x_2$, $\dot{x}_{2\text{-}No.1_i} / \dot{x}_{1\text{-}No.1_i} < -\xi_1$.

Above and below the switching surface S_1 ,there are $\dot{x}_{2\text{-}No.3_{ic}} / \dot{x}_{1\text{-}No.3_{ic}} > -\xi_1$ and $\dot{x}_{2\text{-}No.1_i} / \dot{x}_{1\text{-}No.1_i} < -\xi_1$. Therefore, when the trajectory reaches the switching surface S_1, it reaches the origin along the switching surface S_1.

In summary of the previous analysis, the phase trajectory of the initial point in $No.0_1$ subspace is shown in figure 3.

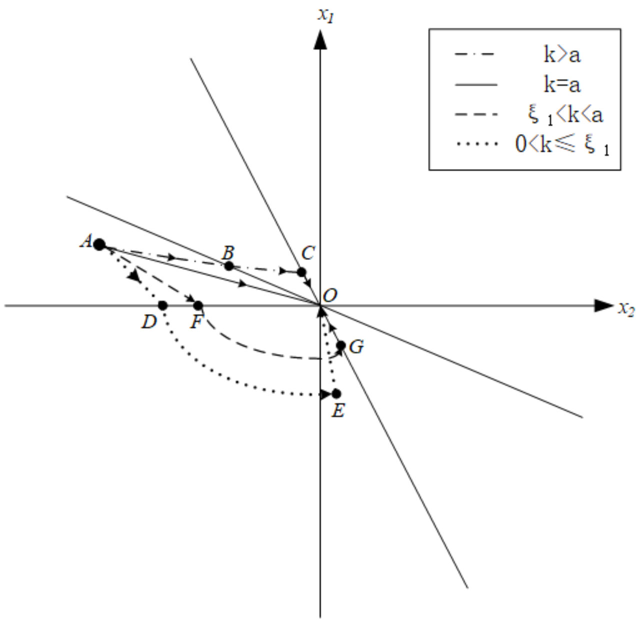

Fig.3. Phase trajectory of initial point in $No.0_{ia}$ subspace

(2). When the initial point is in $No.0_{ib}$ subspace, can get $a<0$.Combined with known conditions, can get $\dot{x}_1<0, \dot{x}_2>0$, the trajectory moves along the negative direction of x_1 axis and the positive direction of x_2 axis. When x_2 is zero, x_1 is still less than zero. Therefore, the point will cross the negative axis of x_1 to the $No.0_{ic}$ subspace. From the previous proof, there are two kinds of trajectories. as shown in the figure 4.

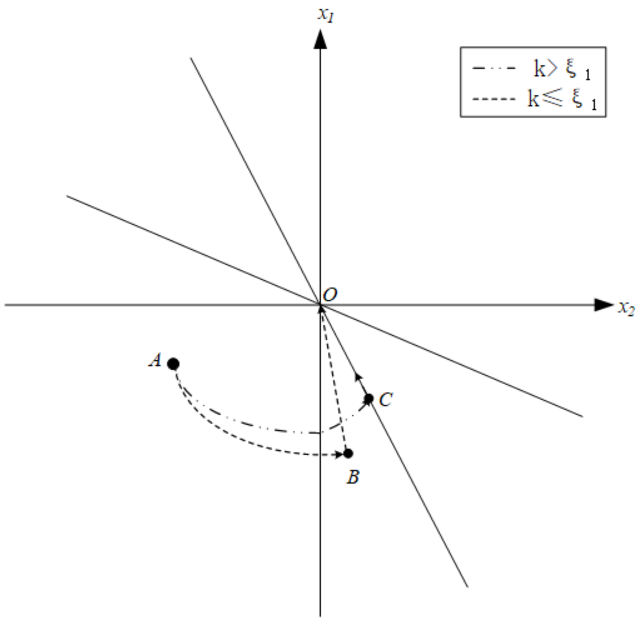

Fig.4. Phase trajectory of initial point in $No.0_{ib}$ subspace

(3).When the initial point is in $No.0_{ic}$ subspace, can get $\xi_2>\xi_1>a$. There are also three situations:

①.The value of k is in interval $(0,a)$, combined with known conditions, can get $\dot{x}_1=x_2>0$, $\dot{x}_2<-ax_2=-a\dot{x}_1$. The decreasing speed of x_2 is less than a times of the increasing speed of x_1. At this time, x_1 is increasing function and the trajectory is always below partition surface $x_2+ax_1=0$. With the increase of x_1, there must be a point (x_2, x_1) that satisfies $x_2+kx_1=0$. At this point, there are $\dot{x}_2=-kx_2-\omega_{01}^{-1}\omega_{02}(x_2+kx_1)=-kx_2=-k\dot{x}_1$ and $x_2+kx_1=0$. Similar to theorem 1, the trajectory reaches the origin along line $x_2+kx_1=0$.

②.When $k=a$, it conforms to theorem 1 and the trajectory will reaches the origin along the partition surface $x_2+ax_1=0$.

③.The value of k is in interval $(a,+\infty)$, combined with known conditions, can get x_1 is increasing function and the trajectory is always above partition surface $x_2+ax_1=0$. With the increase of x_1, the trajectory will reach the switching surface S_1. From the previous proof, it can be concluded that the phase trajectory will reach the origin along the switching surface S_1.

In summary of the previous analysis, the phase trajectory of the initial point in $No.0_{ic}$ subspace is shown in figure 5.

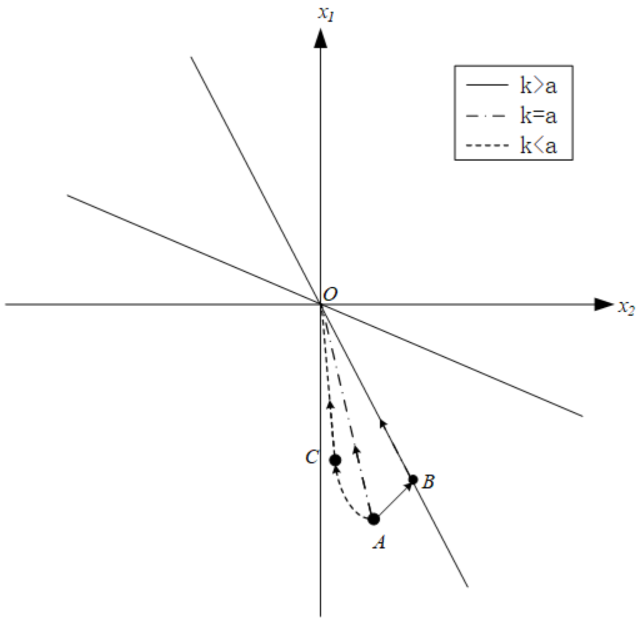

Fig.5. Phase trajectory of initial point in $No.0_{ic}$ subspace

3.2 Trajectory analysis of $No.1_i$ subspace

When the initial point is in $No.1_i$ subspace, can get $\omega_{11}<0$, there are three situations:

①.The value of k is in interval $(0,a)$, there are $x_2+kx_1<0$, $\omega_{12}<0$ or $\omega_{12}>0$, $\omega_{11}^{-1}\omega_{12}<\xi_1<a$.

Partial derivative of m_1 to k, the $Eq.(33)$ can be obtained:

$$\frac{\partial m_1}{\partial k}=-\left(x_2+\omega_{11}^{-1}\omega_{12}x_1\right) \tag{33}$$

It is easy to get that $\partial m_1/\partial k<0$, m_1 is an decreasing function. The maximum value is:

$$\lim_{k\to a} m_1=\lim_{k\to a}\left(-kx_2-\omega_{11}^{-1}\omega_{12}\left(x_2-kx_1\right)\right)=-ax_2 \tag{34}$$

Where $k<a$,can get $\dot{x}_2<-ax_2=-a\dot{x}_1$. The decreasing speed of x_2 is less than a times of the increasing speed of x_1.At this time, the trajectory is always above the partition surface $x_2+ax_1=0$, and must reach the switching surface S_2.

As the trajectory reaches the switching surface S_2, there are $No.3_{ia}$ subspace and $No.1_i$ subspace above and below the switching surface and the $Eq.(35)$ can be obtained

$$\begin{cases} \dot{x}_{2\text{-}No.3_{ia}} = -kx_2 - \Omega_{31}^{-1}\Omega_{32}\left(x_2 + kx_1\right) \\ \dot{x}_{2\text{-}No.1_i} = -kx_2 - \Omega_{11}^{-1}\Omega_{12}\left(x_2 + kx_1\right) \end{cases} \quad (35)$$

Combined $\Omega_{31}^{-1}\Omega_{32} > \Omega_{21}^{-1}\Omega_{22}$, $\Omega_{21}^{-1}\Omega_{22}=\Omega_{11}^{-1}\Omega_{12}$, $x_2 + kx_1 < 0$ and $Eq.(35)$, it is easy to get $\dot{x}_{2\text{-}No.3_{ia}} > \dot{x}_{2\text{-}No.1_i}$ .Therefore, the trajectory must pass through the switching surface S_2 and become more gentler. After that, the trajectory is carried out according to the operation law of $No.3_i$ subspace.

②.When $k = a$, it conforms to theorem 1. The trajectory reaches the origin along the partition surface $x_2 + ax_1=0$.

③.The value of k is in interval $(a,+\infty)$ .it is easy to get $\dot{x}_2 > -ax_2 = -a\dot{x}_1$. The decreasing speed of x_2 is more than a times of the increasing speed of x_1 .At this time, the trajectory is always below the partition surface $x_2 + ax_1=0$, and must reach the switching surface S_1 . From the previous proof, the trajectory reaches the origin along the switching surface S_1 .

In summary of the previous analysis, the phase trajectory of the initial point in $No.1_i$ subspace is shown in figure 6.

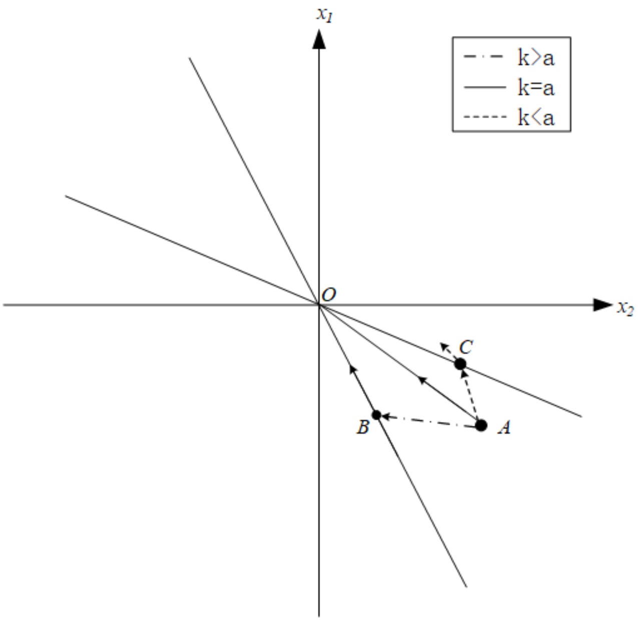

Fig.6. Phase trajectory of initial point in $No.1_i$ subspace

3.3 Trajectory analysis of $No.2_i$ subspace

The initial point is in $No.2_i$ subspace, the trajectory is symmetric with respect to the origin with respect to the phase trajectory of the initial point at $No.1_i$ subspace. With the increase of k value, there are three situations:

①.The value of k is in the interval $(0,a)$, the trajectory must reach the switching surface S_2 , and pass through the switching surface S_2 to the $No.0_i$ subspace. After that, the trajectory is carried out according to the operation law of $No.0_i$ subspace.

②.When $k = a$, it conforms to theorem 1. The trajectory reaches the origin along the partition surface $x_2 + ax_1=0$.

③.The value of k is in the interval $(a,+\infty)$, the trajectory must reach the switching surface S_1 and then reach the origin along the switching surface S_1 .

In summary of the previous analysis, the phase trajectory of the initial point in $No.2_i$ subspace is shown in figure 7.

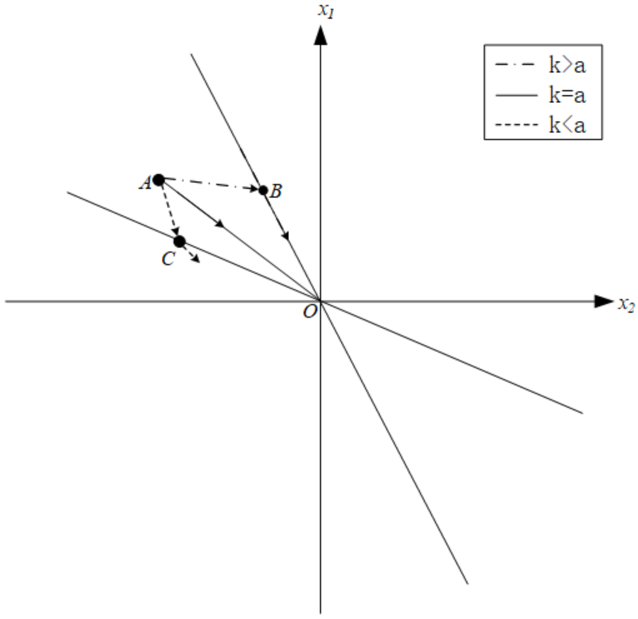

Fig.7. Phase trajectory of initial point in $No.2_i$ subspace

3.4 Trajectory analysis of $No.3_i$ subspace

The initial point is in $No.3_i$ subspace, the trajectory is symmetric with respect to the origin with respect to the phase trajectory of the initial point in $No.0_i$ subspace.

(1)When the initial point is in $No.3_{ia}$ subspace, it is easy to get initial point moves along the positive direction of x_1 axis and the negative direction of x_2 axis. With the increase of k value, there are three situations:

①.The value of k is in interval $(0,a)$, the trajectory is always above partition surface $x_2 + ax_1=0$, when x_1 is zero, x_2 is still greater than zero. At this time, the point will cross the positive axis of x_2 to the $No.3_{ib}$ subspace.

As the trajectory moves to $No.3_{ib}$ subspace. It is easy to get the trajectory moves along the positive direction of x_1 axis and the negative direction of x_2 axis. When x_2 is zero, x_1 is still greater than zero. At this time, the point will cross the positive axis of x_1 to the $No.3_{ic}$ subspace.

As the trajectory moves to $No.3_{ic}$ subspace, there are two situations:

When the value of k is in interval $(0,\xi_1]$, with the decrease of x_1 , there must be a point (x_2, x_1) that satisfies $x_2 + kx_1=0$. The trajectory will reach the origin along line $x_2 + kx_1=0$.

When the value of k is in interval (ξ_1,a), with the increase of x_1 , the trajectory must reach the switching surface S_1 and then reach the origin along the switching surface S_1 .

②.When $k = a$, it conforms to theorem 1. The trajectory reaches the origin along the partition surface $x_2 + ax_1=0$.

③.The value of k is in interval $(a,+\infty)$, the trajectory is always above the partition surface $x_2 + ax_1=0$, and must reach the switching surface S_2 . After that, the trajectory will surely pass through the switching surface S_2 to the switching surface S_1 . From the previous proof, the trajectory reaches the origin along the switching surface S_1 .

In summary of the previous analysis, the phase trajectory of the initial point in $No.3_{ia}$ subspace is shown in figure 8.

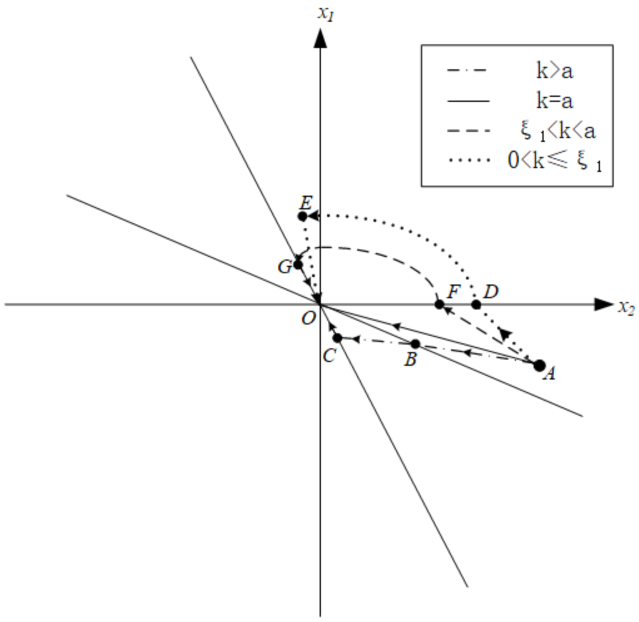

Fig.8. Phase trajectory of initial point in $No.3_{ia}$ subspace

(2).When the initial point is in $No.3_{ib}$ subspace, can get $a<0$.Combined with known conditions, can get $\dot{x}_1>0, \dot{x}_2<0$, the trajectory moves along the positive direction of x_1 axis and the negative direction of x_2 axis. When x_2 is zero, x_1 is still more than zero. Therefore, the point will cross the positive axis of x_1 to the $No.3_{ic}$ subspace. From the previous proof, there are two kinds of trajectories. as shown in the figure 9.

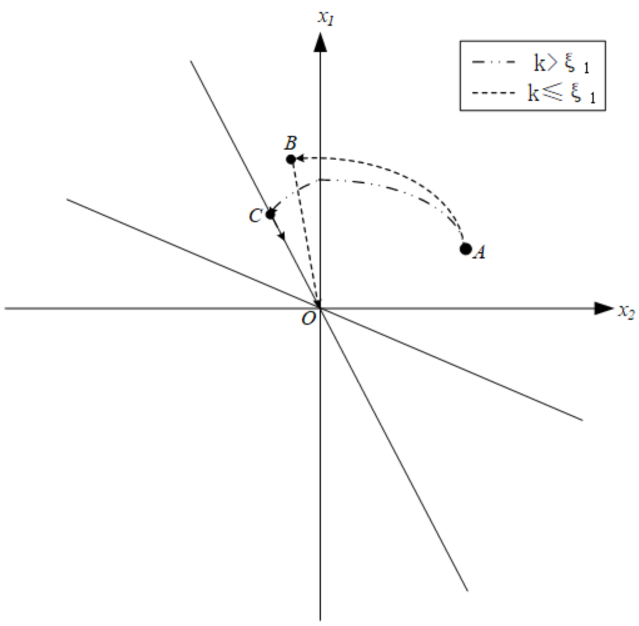

Fig.9. Phase trajectory of initial point in $No.3_{ib}$ subspace

(3).When the initial point is in $No.3_{ic}$ subspace, can get $\xi_2>\xi_1>a$. There are also three situations:

①.The value of k is in interval $(0,a)$, it is easy to get x_1 is decreasing function, and the trajectory is always above the partition surface $x_2+ax_1=0$. With the decrease of x_1, there must be a point (x_2,x_1) that satisfies $x_2+kx_1=0$. The trajectory reaches the origin along line $x_2+kx_1=0$.

②.When $k=a$, it conforms to theorem 1. The trajectory reaches the origin along the partition surface $x_2+ax_1=0$.

③. The value of k is in interval $(a,+\infty)$, with the decrease of x_1, the trajectory must reach the switching surface S_1 and then reach the origin along the switching surface S_1.

In summary of the previous analysis, the phase trajectory of the initial point in $No.3_{ic}$ subspace is shown in figure 10.

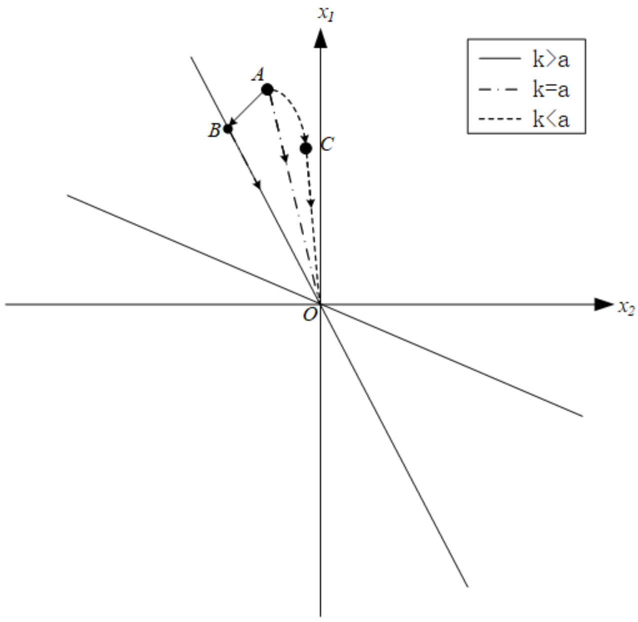

Fig.10. Phase trajectory of initial point in $No.3_{ic}$ subspace

4.Simulation

Considering the following nonlinear system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(\boldsymbol{x}) + g(\boldsymbol{x}) \cdot u + d \end{cases} \tag{36}$$

Where $\boldsymbol{x} \in [x_1, x_2]^T, u \in R, d = 0.1 \times \cos(\mathrm{t})$, $g(\boldsymbol{x}) = \dfrac{\cos x_1}{7.3 - 1.5\cos^2 x_1}$ and $f(\boldsymbol{x}) = \dfrac{107.9\sin x_1 - 1.5x_2^2 \cos x_1 \sin x_1}{7.3 - 1.5\cos^2 x_1}$.

The switching surfaces are chosen as follows:

$$\begin{cases} S_1 = x_2 + x_1 = 0 \\ S_2 = x_2 + 5x_1 = 0 \end{cases} \tag{37}$$

Four appropriate point P_1, P_2, P_3, P_4 are chosen on the switching surfaces, they are $P_1(4,-4)$, $P_2(5.4545,-1.0909)$, $P_3(-4,4)$, $P_4(-5.4545,1.0909)$. The UASs can be obtained:

$$\begin{cases} h_0 = 0.2917x_2 + 0.5417x_1 + 1 \\ h_1 = -0.1667x_2 + 0.0833x_1 + 1 \\ h_2 = 0.1667x_2 - 0.0833x_1 + 1 \\ h_3 = -0.2917x_2 - 0.5417x_1 + 1 \end{cases} \tag{38}$$

Where

$$\omega_1 = \begin{cases} 0.2917 & S_1 < 0, S_2 < 0 \\ -0.1667 & S_1 \geq 0, S_2 < 0 \\ 0.1667 & S_1 < 0, S_2 \geq 0 \\ -0.2917 & S_1 \geq 0, S_2 \geq 0 \end{cases}$$

$$\omega_2 = \begin{cases} 0.5417 & S_1 < 0, S_2 < 0 \\ 0.0833 & S_1 \geq 0, S_2 < 0 \\ -0.0833 & S_1 < 0, S_2 \geq 0 \\ -0.5417 & S_1 \geq 0, S_2 \geq 0 \end{cases}$$

The initial point P and the value of k are shown in Table 1.

Tab.1. Initial values and the value of k

$(x_2(0), x_1(0))$	$0<k<\xi_1$	$\xi_1<k<a$	$k=a$	$k>a$
(-8,1.2)	0.5	2	20/3	10
(8,-1.2)	0.5	2	20/3	10
$(x_2(0), x_1(0))$	$0<k<a$		$k=a$	$k>a$
(4,-8)	0.1		0.5	4
(8,-4)	1		2	5
(-4,8)	0.1		0.5	4
(-8,4)	1		2	5
$(x_2(0), x_1(0))$	$0<k<\xi_1$		$k>\xi_1$	
(-5,-5)	0.2		5	
(5,5)	0.2		5	

Approaching law is designed as $N_i = k(w - h_i)$. The control input $u = g^{-1}(\mathbf{x})\left(-f(\mathbf{x}) + \omega_{i1}^{-1}(-\omega_{i2}x_2 + N_i)\right)$.

The simulation of phase trajectory with initial point in $No.0_i$ is shown in $Fig.11$ to $Fig.13$. The simulation of phase trajectory with initial point in $No.1_i$ is shown in $Fig.14$. The simulation of phase trajectory with initial point in $No.2_i$ is shown in $Fig.15$ .The simulation of phase trajectory with initial point in $No.3_i$ is shown in $Fig.16$ to $Fig.18$.

From $Fig.11$ to $Fig.18$,it can be seen that simulation of initial point and k value is coincidence previous proof. When $k < \xi_1$ or $k=a$, the phase trajectory will reaches the origin along line $x_2 + kx_1 = 0$ or partition surface $x_2 + ax_1 = 0$,and the trajectory will not chattering on switching surfaces. However，in this case, the convergence rate will become slower. While $k > \xi_1$, the trajectory will reaches and chattering on switching surfaces S_1 . Therefore, the k-value of the approaching law can be taken as ξ_1 or a , and ξ_1 can be taken as a larger value to speed up the convergence.

From theoretical analysis and simulation experiments, it can be found that the purpose of switching surface S_1 is sliding surface, and the purpose of switching surface S_2 is to accelerate the convergence rate of phase trajectory.

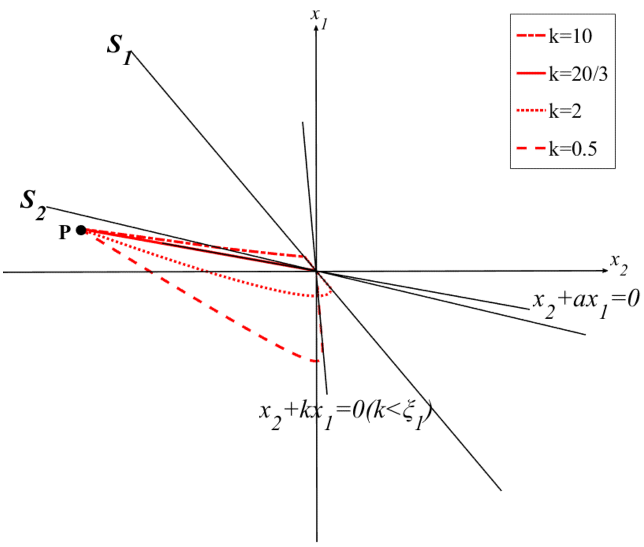

Fig.11. initial point in $No.0_{ia}$ subspace

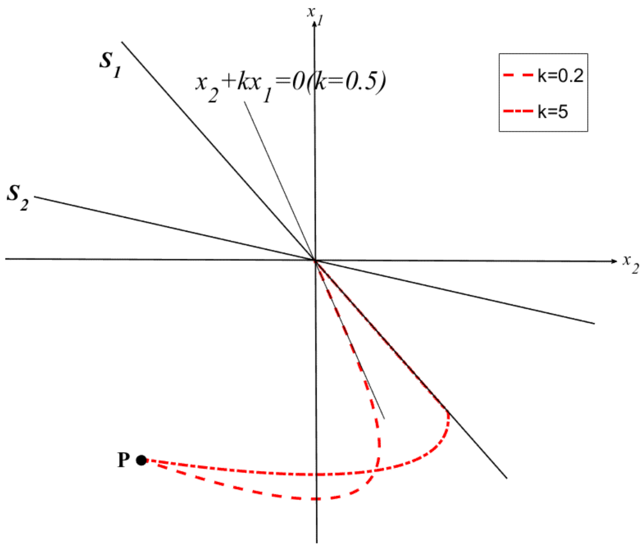

Fig.12. initial point in $No.0_{ib}$ subspace

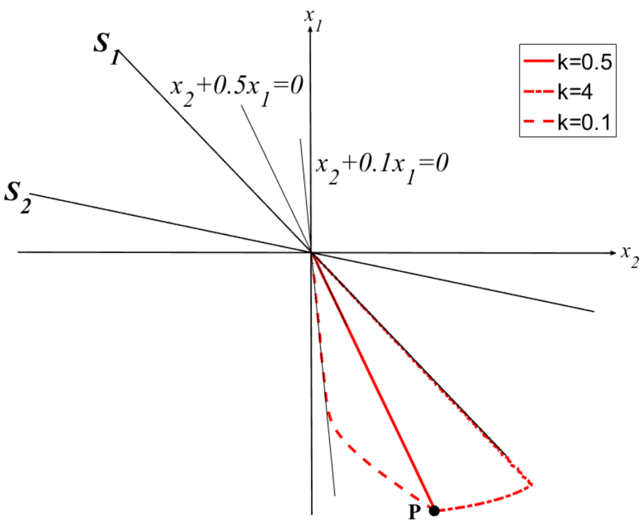

Fig.13. initial point in $No.0_{ic}$ subspace

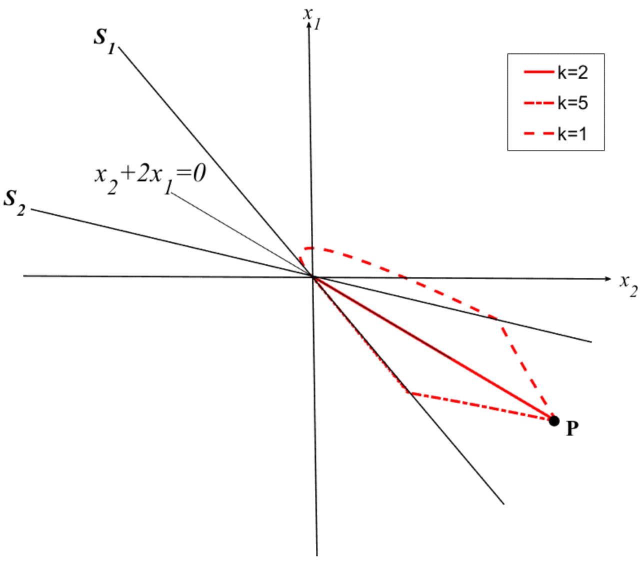

Fig.14. initial point in $No.1_i$ subspace

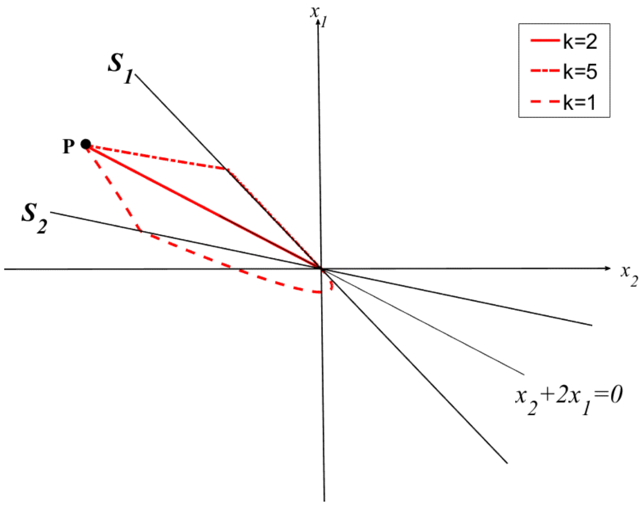

Fig.15. initial point in No.2_i subspace

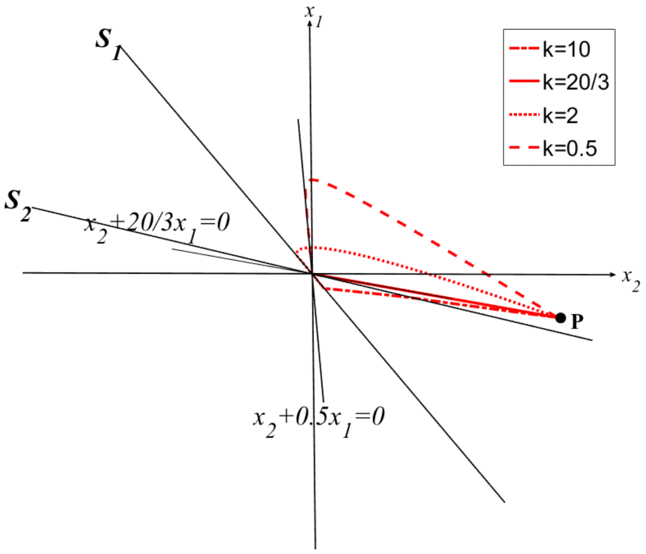

Fig.16. initial point in *No.*3_{ia} subspace

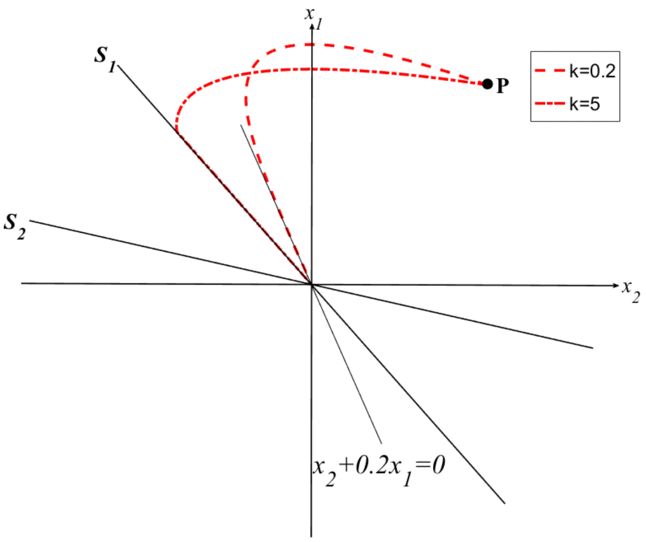

Fig.17. initial point in *No.*3_{ib} subspace

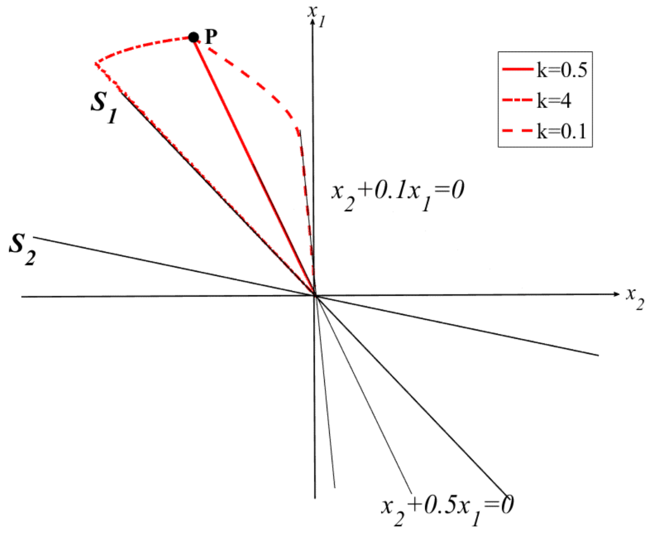

Fig.18. initial point in *No.*3_{ic} subspace

5.Conclusion

In this paper, the phase trajectory analysis of the closed-loop system with UAS-SMC is given for different initial points and k-values of approaching law. According to the theoretical proof and the simulation experiment, the operation of the phase trajectory is related to the initial points and k-value of the system, and the phase trajectory will be carried out according to certain rule. According to this law, under the condition that the initial point of the system is known, the phase trajectory of the system state can be predicted. In addition, it is found that the purpose of switching surface S_1 is sliding surface, and the purpose of switching surface S_2 is to accelerate the convergence rate of phase trajectory. At the same time, it is also obtained that the chattering phenomenon can be avoided by selecting the appropriate k-value of approaching law. In the next step, we can design other forms of switching surface to replace switching surface S_2 , to improve the robustness and rapidity of sliding mode control with unidirectional auxiliary surfaces method.

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References

Yang, J, et al , 2012. Sliding mode variable structure control theory: A survey. 31th Chinese Control Conference. 3197-3202.

Utkin, V., 1977. Variable structure systems with sliding modes. IEEE Transactions on Automatic Control. 22,212-222.

Cheema, M.A.M., et al , 2017. Combined speed and direct thrust force control of linear permanent-magnet synchronous motors with sensorless speed estimation using a sliding-mode control with integral action. IEEE Transactions on Industrial Electronics. 64, 3489-3501.

Pukdeboon C., et al , 2009. Optimal sliding mode controllers for attitude tracking of spacecraft. 2009 IEEE International Conference on Control Applications.

Tapan, S., et al, 2018. Vibration control of a hybrid magnetic bearing using an adaptive sliding mode technique. Journal of Vibration and Control. 24,1848-1860.

Luo, X., et al, 2018. Time Delay Estimation-based Adaptive Sliding-Mode Control for Nonholonomic Mobile Robots. International Journal of Applied Mathematics in Control Engineering. 1(1), 1-8.

Chen, Q., et al, 2018. Finite Time Synergetic Control for Quadrotor UAV with Disturbance Compensation. International Journal of Applied Mathematics in Control Engineering. 1(1), 31-38.

Fu J, et al, 2011. Robust Sliding Mode Positively Invariant Set for Nonlinear Continuous System. Acta Automatica Sinica. 11, 1395-1401.

Hu T W, et al, 2013. Exponential Approach Law for Sliding Mode Control with Unidirectional Auxiliary Surfaces. Electronics Optics & Control. 7, 36-41.

Zhang P, et al , 2018. Double power reaching law for sliding mode control with unidirectional auxiliary surfaces. 36th Chinese Control Conference. 3798-3803.

Fu J, et al, 2011. Robust sliding mode control with unidirectional auxiliary surfaces for nonlinear system with state constraints. Control and Decision. 9,1288-1294.

Ren Z X, et al, 2013. Sliding Mode Flight Control with Unidirectional Auxiliary Surfaces Based on NDO. Electronics Optics & Control. 9, 48-52.

He N B, et al, 2015.Terminal Sliding Mode Control with Unidirectional Auxiliary Surfaces for Hypersonic Vehicles Based on Adaptive Disturbance Observer. Mathematical Problems in Engineering. 285716, 1-11.

Liu S, et al, 2018. Unidirectional sliding mode control for six-phase PMSM system. Electric Machines and Control. 22, 10-19.

Yang, Z., et al., 2018. Back-stepping Sliding Mode with Unidirectional Auxiliary Surfaces for HSV with Attitude Constrains. International Journal of Applied Mathematics in Control Engineering. 1(2), 187-193.

Fu J, et al, 2013. Chattering-free condition for sliding mode control with unidirectional auxiliary surfaces. Transactions of the Institute of Measurement and Control. 35, 593-605.

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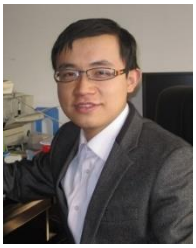

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