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Multi-Attribute Group Decision-Making Methods Based on Possibility Hesitant Fuzzy *N*-Soft Sets

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ABSTRACT

This paper defines a novel structure called possibility hesitant fuzzy *N*-Soft sets (*PHFNSSs*) by reasonable combination of hesitant fuzzy *N*-Soft sets and possibility soft sets. Possibility fuzzy soft sets where a possibility of each alternative in the universe is attached with the parameterization of fuzzy sets while defining a fuzzy soft set. *HFNSSs* presents the importance of grades and the reasonable of hesitancy in real decision-makings. So the new model account for the situations when a decision maker hesitates to provide multiple estimates of an object and considers the possibility degree of belongingness of the alternatives. Moreover, some basic and useful operations and properties are given. Finally, we extends this method TOPSIS so that it can operate under *PHFNSSs* information in practical application.

1. Introduction

In practical decision-making problems, experts sometimes take a hesitant and ambiguous attitude. To correct this shortcoming, hesitancy was combined with other mathematical structures. The research about hesitant fuzzy sets were a generalization of fuzzy sets proposed by V.Torra (V.Torra et al., 2009). The emergence of soft set theory provides us with a parameterized method to solve life uncertainty problems by Molodtsov in (Molodtsov D et al., 1999). Maji et al. (Maji PK et al., 2001) proposed fuzzy soft set by combining the soft set and fuzzy set. Alkhazaleh et al. (S.Alkhazaleh et al., 2001) defined the concept of possibility fuzzy soft set and gave its applications in decision making and medical diagnosis. The soft set was in a binary environment whose values were 0 or 1. But the data in real life was non-binary essentially. Motivated by these problems, Fatimach et al. (Fatimah, F et al., 2018) create the improved soft set model called *N*-soft set which is a new research orientation and the importance of hierarchy was emphasized. For flexibility in decision making, a novel hybrid structure hesitant fuzzy *N*-soft sets (henceforth, *HFNSSs*) (Akram M et al., 2018) made up of *N*-soft sets and hesitant fuzzy soft set. And they combined *N*-soft sets with other mathematical model and got interval valued hesitant fuzzy *N*-soft sets, neutrosophic vague *N*-soft sets, generalized vague *N*-soft sets and so on (e.g. Akram M et al., 2019 and J.B. Liu et al., 2020 and Y.N. Chen et al., 2020). For decision-making, the Choice and L-choice value for *HFNSSs* (Akram M et al., 2019), one of the best known methods was TOPSIS introduced by Hwang and Yoon. Later, a lot of improved TOPSIS methods were proposed such as fuzzy TOPSIS, intuitionistic fuzzy TOPSIS, Pythagorean fuzzy TOPSIS and interval-valued fuzzy TOPSIS (e.g.S.Nadaban et al., 2016 and F. E. Boran et al., 2009 and A.Biswas et al., 2019 and T.Y. Chen et al., 2008). TOPSIS is still being the hot spot of both applications(F.J. Esterlla et al., 2017 and G.K. Koulinas et al., 2019) and theoretical implementations.(A.Biswas et al., 2019 and Z.Pei et al., 2019) There are other decision-making methods in (e.g. P.Liu, 2018 and Li, S.et al., 2018 and Li, C.et al., 2019 and Yin, X.et al., 2019)

The paper is organized as five sections. Section 2 provides the relevant theoretical background. In Section 3, we introduce our new hybrid model possibility hesitant fuzzy *N*-Soft sets by reasonable combination of hesitant fuzzy *N*-Soft sets and possibility soft sets and give some basic operations and properties containing equal, inclusion, complement, union, intersection, AND and OR operations of two *PHFNSSs*. Section 4 extends this method TOPSIS so that it can operate under *PHFNSSs* information and gives a practical application. We present our conclusion in Section 5.

2. PRELIMINARIES

In this section, some basic definitions and examples were recapitulated, which is serviceable and appropriate to understand the

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new concepts to be proposed. They are stated as follows:

Definition 2.1 (Fatimah, F et al., 2018) Let U denote the universe of alternatives an E the set of attributes, $T \subseteq E$ .Let $M=\{0,1,2,\cdots,N-1\}$ be the set of ordered grades where $N \in \{2,3,\cdots\}$. A triple (R,T,N) is called an *N*-soft set on U if F is a mapping: $R:T \to 2^{U\times M}$, with the property that for each $t \in T$ and $u \in U$ there exists a unique $(u,g_t) \in U \times M$ such that $(u,g_t) \in R(t)$, $g_t \in M$.

Definition 2.2 (Akram M et al., 2019) Let U denote the universe of alternatives and E the set of attributes. Let $M=\{0,1,2,\cdots,N-1\}$ be the set of ordered grades where $N \in \{2,3,\cdots\}$. A pair (ξ,K) is called fuzzy *N*-soft set, $K=(R,T,N)$ is an N-soft set on U with $N \in \{2,3,\cdots\}$, and ξ is a mapping: $\xi : T \to \bigcup_{t\in T} \Re(R(t))$, such that $\xi(t) \in \Re(R(t))$ for each $t \in T$.

Definition 2.3 (V.Torra et al., 2009) Let U denote the universe of alternatives, then a hesitant fuzzy set on U is defined in terms of a function h that when applied to U returns a subset of $[0,1]$ (i.e. an element from $P([0,1])$).

$$h_J = \{\langle u, h_J(u)\rangle \mid u \in U\}$$

where $h_J(u)$ is a set of some different values in $[0,1]$, representing the possible membership degrees of the element $u \in U$ to the set J , and called $h_J(u)$ a hesitant fuzzy element of HFS.

Definition 2.4 (Akram M et al., 2019) Let U denote the universe of alternatives and the set of attribute E , $T \subseteq E$. A triple $(\hbar_f, T, N)$ is called hesitant fuzzy *N*-soft set (*HFNSS*), when $\hbar_f$ is a mapping defined as

$$\hbar_f : U \times T \to M \times P^*([0,1])$$

when $\hbar_f(u,t)=(g,\hbar_{f_t}(u))$ we interpret that $\hbar_f(u)$ is a non-empty set formed by values in $[0,1]$.

Definition 2.5 (S.Alkhazaleh et al., 2001)

Let $U=\{u_1,\cdots u_i,\cdots u_n\}$ be the universe of alternatives, let E be the set of attributes, $T \subseteq E$, $T=\{e_1,\cdots e_j,\cdots e_m\}$. The pair (U,T) will be called a soft universe. Let $F:T \to I^U$ and μ be a fuzzy subset of T , $\mu : T \to I^U$.

Let $F_\mu : T \to I^U \times I^U$ be a function defined as follows:

$$F_\mu(e) = (F(e)(u), \mu(e)(u)), \forall u \in U$$

Then F_μ is called a possibility fuzzy soft set (PFSS in short) over the U . For each $e_j \in T$, $u_i \in U$,

$$F_\mu(e_j)=\left\{\begin{array}{l}\left(\dfrac{u_1}{F(e_j)(u_1)},\mu(e_j)(u_1)\right)\\ \cdots \\ \left(\dfrac{u_i}{F(e_j)(u_i)},\mu(e_j)(u_i)\right)\\ \cdots \\ \left(\dfrac{u_n}{F(e_j)(u_n)},\mu(e_j)(u_n)\right)\end{array}\right\}.$$

3. Possibility Hesitant fuzzy *N*-Soft sets

In this subsection, we define our extension of the above concepts and propose some fundamental properties and operations for the new structure.

3.1 The concept of possibility hesitant fuzzy N-Soft sets(PHFNSSs)

Definition 3.1 Let $U=\{u_1,\cdots u_i,\cdots u_n\}$ denote the universe of alternatives, let E the s $T \subseteq E$ et of attributes, $T \subseteq E$, $T=\{e_1,\cdots e_j,\cdots e_m\}$. A triple $(\hbar_{f\mu}, T, N)$ is called possibility hesitant fuzzy *N*-soft set, there are three mappings:

$$\hbar_f : U \times T \to M \times P^*([0,1]),\quad \mu : T \to I^U$$

and $\hbar_{f_\mu} : T \to I^{U\times M} \times P^*([0,1])$. That is to say: for each $e_j \in T$, $u_i \in U$,

$$\hbar_{f_\mu}(e_j)=\left\{\begin{array}{l}\left(\dfrac{(u_1,g_1)}{\hbar_f(e_j)(u_1)},\mu(e_j)(u_1)\right)\\ \cdots \\ \left(\dfrac{(u_i,g_i)}{\hbar_f(e_j)(u_i)},\mu(e_j)(u_i)\right)\\ \cdots \\ \left(\dfrac{(u_n,g_n)}{\hbar_f(e_j)(u_n)},\mu(e_j)(u_n)\right)\end{array}\right\}.$$

Here for each parameter e_j and alternative, *PHFNSSs* demonstrated the not only the degree of belongingness of the elements of U in $\hbar_{f_\mu}(e_j)$ but also the degree of possibility of possibility of such belongingness which is represented by $\mu(e_j)$. Meanwhile, the grade is determined by the score

$$\varsigma_T(u_i) = \frac{1}{|k|}\sum_{\alpha\in k}\alpha.$$

$$\varsigma_{e_j}(u_i) = \frac{1}{\left|\hbar_f(e_j)(u_i)\right|}\sum_{\alpha\in\hbar_f(e_j)(u_i)}\alpha \geq \varsigma_{e_j}(u_l) = \frac{1}{\left|\hbar_f(e_j)(u_l)\right|}\sum_{\alpha\in\hbar_f(e_j)(u_l)}\alpha,$$

Where k is a finite hesitant fuzzy element, $|k|$ is the number of elements in k, $T=\{e_1,\cdots e_j,\cdots e_m\}$.

We will rate the objects under discussion based on each attribute. If each $u_i,u_l\in U$, then $g_{ij}\geq g_{il}$.

Example 3.1 A company makes an open tender, there were three non-specific suppliers $U=\{u_1,u_2,u_3\}$ participating in the tender, The successful bidder is selected from all the bidding suppliers according to the pre-determined attributes $T=\{e_1,e_2,e_3\}$. A $(\hbar_{f\mu}|_1,T,3)$ can be obtained as follows:

$$\hbar_{f_\mu}|_1(e_1)=\left\{\begin{array}{l}\left(\frac{(u_1,1)}{\{0.30,0.40\}},0.35\right)\\\left(\frac{(u_2,2)}{\{0.30,0.40,0.50\}},0.40\right)\\\left(\frac{(u_3,0)}{\{0.30\}},0.30\right)\end{array}\right\};$$

$$\hbar_{f_\mu}|_1(e_2)=\left\{\begin{array}{l}\left(\frac{(u_1,2)}{\{0.70,0.80\}},0.75\right)\\\left(\frac{(u_2,0)}{\{0.40,0.60\}},0.50\right)\\\left(\frac{(u_3,1)}{\{0.30,0.50,0.80\}},0.50\right)\end{array}\right\};$$

$$\hbar_{f_\mu}|_1(e_3)=\left\{\begin{array}{l}\left(\frac{(u_1,1)}{\{0.10,0.15,0.20\}},0.30\right)\\\left(\frac{(u_2,1)}{\{0.10,0.20\}},0.20\right)\\\left(\frac{(u_3,0)}{\{0.10\}},0.10\right)\end{array}\right\}.$$

3.2 The operations and properties of PHFNSSs

Definition 3.2 Let $(\hbar_{f\mu}|_1,T,N_1)$ and $(\hbar_{f\mu}|_2,S,N_2)$ be two *PHFNSSs* of *U*. Then $(\hbar_{f\mu}|_1,T,N_1)$ and $(\hbar_{f\mu}|_2,S,N_2)$ are said to equal if and only if $N_1=N_2$, $T=S$, $\hbar_{f_\mu}|_1=\hbar_{f_\mu}|_2$.

Definition 3.3 Let $(\hbar_{f\mu}|_1,T,N_1)$ and $(\hbar_{f\mu}|_2,S,N_2)$ be two *PHFNSSs* of *U*. If $\forall e\in T, T\subseteq S\subseteq E, N_1\leq N_2$, $g_1\leq g_2$. $\hbar_f|_1$ is also a hesitant fuzzy subset of $\hbar_f|_2$ and μ_1 is a possibility fuzzy subset of μ_2, then $(\hbar_{f\mu}|_1,T,N_1)$ is said to be a possibility hesitant fuzzy subset of $(\hbar_{f\mu}|_2,S,N_2)$ denoted by $(\hbar_{f\mu}|_1,T,N_1)\subseteq(\hbar_{f\mu}|_2,S,N_2)$.

Example 3.2 Let $(\hbar_{f\mu}|_2,S,3)$ be another PHF3SS, $S=\{e_1,e_2,e_3,e_4\}$, then $(\hbar_{f\mu}|_2,S,3)$ can be obtained as follows:

$$\hbar_{f_\mu}|_2(e_1)=\left\{\begin{array}{l}\left(\frac{(u_1,1)}{\{0.2,0.30,0.40\}},0.40\right)\\\left(\frac{(u_2,2)}{\{0.30,0.40,0.50\}},0.45\right)\\\left(\frac{(u_3,0)}{\{0.20,0.30\}},0.30\right)\end{array}\right\};$$

$$\hbar_{f_\mu}|_2(e_2)=\left\{\begin{array}{l}\left(\frac{(u_1,2)}{\{0.70,0.80,0.90\}},0.85\right)\\\left(\frac{(u_2,0)}{\{0.40,0.50,0.60\}},0.,60\right)\\\left(\frac{(u_3,1)}{\{0.30,0.50,0.60,0.80\}},0.60\right)\end{array}\right\};$$

Obviously, we can find $(\hbar_{f\mu}|_1,T,3)\subseteq(\hbar_{f\mu}|_2,S,3)$.

$$\hbar_{f_\mu}|_2(e_3)=\left\{\begin{array}{l}\left(\frac{(u_1,1)}{\{0.10,0.15,0.20,0.35\}},0.35\right)\\\left(\frac{(u_2,2)}{\{0.10,0.20,0.40\}},0.25\right)\\\left(\frac{(u_3,0)}{\{0.10,0.20\}},0.15\right)\end{array}\right\};$$

$$\hbar_{f_\mu}|_2(e_4)=\left\{\begin{array}{l}\left(\frac{(u_1,1)}{\{0.60,0.70\}},0.60\right)\\\left(\frac{(u_2,2)}{\{0.50,0.70,0.80\}},0.70\right)\\\left(\frac{(u_3,0)}{\{0.20,0.30\}},0.25\right)\end{array}\right\}.$$

Definition 3.4 A *PHFNSS* is said to be a possibility hesitant absolute fuzzy *N*-soft set denoted by $\tilde{A}_{f_\mu}$, if

$$\tilde{A}_{f_\mu}: T\rightarrow I^{U\times M}\times P^*([0,1]),$$

such that for each $e_j \in T$, $u_i \in U$,

$$\tilde{A}_{f_\mu}(e_j)=\left\{\begin{array}{l}\left(\dfrac{(u_1,1)}{\hbar_f(e_j)(u_1)=\{1\}},1\right)\\ \cdots \\ \left(\dfrac{(u_i,1)}{\hbar_f(e_j)(u_i)=\{1\}},1\right)\\ \cdots \\ \left(\dfrac{(u_n,1)}{\hbar_f(e_j)(u_n)=\{1\}},1\right)\end{array}\right\}.$$

Definition 3.4 A *PHFNSS* is said to be a possibility hesitant absolute fuzzy *N*-soft set denoted by $\tilde{\varphi}_{f_\mu}$,if

$$\tilde{\varphi}_{f_\mu}: T \to I^{U\times M} \times P^*([0,1]),$$

such that for each $e_j \in T$, $u_i \in U$,

$$\tilde{\varphi}_{f_\mu}(e_j)=\left\{\begin{array}{l}\left(\dfrac{(u_1,0)}{\hbar_f(e_j)(u_1)=\{0\}},0\right)\\ \cdots \\ \left(\dfrac{(u_i,0)}{\hbar_f(e_j)(u_i)=\{0\}},0\right)\\ \cdots \\ \left(\dfrac{(u_n,0)}{\hbar_f(e_j)(u_n)=\{0\}},0\right)\end{array}\right\}.$$

Definition 3.5 A complement of a *PHFNSS* $(\hbar_{f\mu}|_1, T, N_1)$ is defined as $(\hbar^c_{f_\mu}|_1, T, N_1)$ that verifies the property $\hbar_{f_\mu}(e)\cap\hbar^c_{f_\mu}(e)=\phi$.

$$(\hbar^c_{f_\mu}|_1, T, N_1)^c \neq (\hbar_{f_\mu}|_1, T, N_1) \qquad (*)$$

Obviously, (*) as the complement of *N*-soft set not only one, so we can get many complement sets of $(\hbar_{f\mu}|_1, T, N_1)$. One of the complement sets is the hesitant fuzzy complement is

$$\hbar^c_f(e)(u)=\bigcup_{k\in\hbar_f(u)}\{1-k\};$$

For all $u \in U$, $e \in T$, the possibility fuzzy complement is

$$\mu^c(e)(u)=1-\mu(e)(u)$$

the grade complement is $g^c(e)(u)=N_1-1-g(e)(u)$.

Example 3.3 $(\hbar^c{}_{f\mu}|_1, T, 3)$ is the complement of $(\hbar_{f\mu}|_1, T, 3)$ from the Example 3.1 as follows:

$$\hbar^c_{f_\mu}|_1(e_1)=\left\{\begin{array}{l}\left(\dfrac{(u_1,1)}{\{0.70,0.60\}},0.65\right)\\ \left(\dfrac{(u_2,0)}{\{0.70,0.60,0.50\}},0.60\right)\\ \left(\dfrac{(u_3,2)}{\{0.70\}},0.70\right)\end{array}\right\};$$

$$\hbar^c_{f_\mu}|_1(e_2)=\left\{\begin{array}{l}\left(\dfrac{(u_1,0)}{\{0.30,0.20\}},0.25\right)\\ \left(\dfrac{(u_2,2)}{\{0.60,0.40\}},0.50\right)\\ \left(\dfrac{(u_3,1)}{\{0.70,0.50,0.20\}},0.50\right)\end{array}\right\};$$

$$\hbar^c_{f_\mu}|_1(e_3)=\left\{\begin{array}{l}\left(\dfrac{(u_1,1)}{\{0.90,0.85,0.80\}},0.70\right)\\ \left(\dfrac{(u_2,1)}{\{0.90,0.80\}},0.80\right)\\ \left(\dfrac{(u_3,2)}{\{0.90\}},0.90\right)\end{array}\right\}.$$

Definition 3.6 Let U be universe of objects and let $(\hbar_{f\mu}|_1, T, N_1)$ and $(\hbar_{f\mu}|_2, S, N_2)$ be two *PHFNSSs*. Then their union is denoted by

$$(\hbar_{f\mu}|_1, T, N_1)\tilde{\cup}(\hbar_{f\mu}|_2, S, N_2)=(\delta_{f_\mu}, T\cup S, \max(N_1,N_2)).$$

(1) $e_j \in T-S: g_{ij}=g_{ij}|_1$, $\delta_{f_\mu}=\hbar_{f_\mu}|_1$.

(2) $e_j \in S-T$: $g_{ij}=g_{ij}|_2$, $\delta_{f_\mu}=\hbar_{f_\mu}|_2$.

(3) $e_j \in T\cap S$

$$g_{ij}=\max(g_{ij}|_1, g_{ij}|_2),$$
$$\delta_{j_\mu}=\max(\hbar_{f_\mu}|_1, \hbar_{f_\mu}|_2).$$

Definition 3.7 Let U be universe of objects, and let $(\hbar_{f\mu}|_1, T, N_1)$ and $(\hbar_{f\mu}|_2, S, N_2)$ be two *PHFNSSs*. Then their intersection is denoted by

$$(\hbar_{f\mu}|_1, T, N_1)\tilde{\cap}(\hbar_{f\mu}|_2, S, N_2)=(\tau_{f\mu}, T\cap S, \min(N_1,N_2)).$$

$$\forall e_j \in T\cap S: g_{ij}=\min(g_{ij}|_1, g_{ij}|_2),\ \delta_{j_\mu}=\min(\hbar_{f_\mu}|_1, \hbar_{f_\mu}|_2).$$

Example 3.4 We consider the two PHF3SSs, $(\hbar_{f\mu}|_1, T, 3)$ and $(\hbar_{f\mu}|_2, S, 3)$ defined in Example 3.1 and 3.2, respectively.

$$(\hbar_{f\mu}|_1, T, 3)\tilde{\cap}(\hbar_{f\mu}|_2, S, 3)=(\hbar_{f\mu}|_1, T, 3),$$
$$(\hbar_{f\mu}|_1, T, 3)\tilde{\cup}(\hbar_{f\mu}|_2, S, 3)=(\hbar_{f\mu}|_2, S, 3).$$

Proposition 3.1 Let $(\hbar_{f\mu}|_1, T, N)$ be a *PHFNSS*, $\tilde{A}_{f_\mu}$ be a possibility possibility hesitant absolute fuzzy N-soft set, $\tilde{\varphi}_{f_\mu}$ a possibility hesitant absolute fuzzy N-soft set. Suppose that they have the same parameter set T, then:

(1) $\left(\hbar_{f\mu}|_1,T,N\right)\tilde{\cup}\tilde{\varphi}_{f_\mu}=\left(\hbar_{f\mu}|_1,T,N\right)$;

(2) $\left(\hbar_{f\mu}|_1,T,N\right)\tilde{\cap}\tilde{\varphi}_{f_\mu}=\tilde{\varphi}_{f_\mu}$;

(3) $\left(\hbar_{f\mu}|_1,T,N\right)\tilde{\cup}\tilde{A}_{f_\mu}=\tilde{A}_{f_\mu}$;

(4) $\left(\hbar_{f\mu}|_1,T,N\right)\tilde{\cap}\tilde{A}_{f_\mu}=\left(\hbar_{f\mu}|_1,T,N\right)$.

Proposition 3.2 Let $\left(\hbar_{f\mu}|_1,T,N\right)$, $\left(\hbar_{f\mu}|_2,S,N\right)$, $\left(\hbar_{f\mu}|_3,M,N\right)$ be three *PHFNSSs*, then:

(1) $\left(\hbar_{f\mu}|_1,T,N\right)\tilde{\cup}\left(\hbar_{f\mu}|_2,S,N\right)=\left(\hbar_{f\mu}|_2,S,N\right)\tilde{\cup}\left(\hbar_{f\mu}|_1,T,N\right)$;

(2) $\left(\hbar_{f\mu}|_1,T,N\right)\tilde{\cap}\left(\hbar_{f\mu}|_2,S,N\right)=\left(\hbar_{f\mu}|_2,S,N\right)\tilde{\cap}\left(\hbar_{f\mu}|_1,T,N\right)$;

(3) $$\begin{aligned}&\left(\hbar_{f\mu}|_1,T,N\right)\tilde{\cup}\left(\left(\hbar_{f\mu}|_2,S,N\right)\tilde{\cup}\left(\hbar_{f\mu}|_3,M,N\right)\right)\\&=\left(\left(\hbar_{f\mu}|_2,S,N\right)\tilde{\cup}\left(\hbar_{f\mu}|_1,T,N\right)\right)\tilde{\cup}\left(\hbar_{f\mu}|_3,M,N\right);\end{aligned}$$

(4) $$\begin{aligned}&\left(\hbar_{f\mu}|_1,T,N\right)\tilde{\cap}\left(\left(\hbar_{f\mu}|_2,S,N\right)\tilde{\cap}\left(\hbar_{f\mu}|_3,M,N\right)\right)\\&=\left(\left(\hbar_{f\mu}|_2,S,N\right)\tilde{\cap}\left(\hbar_{f\mu}|_1,T,N\right)\right)\tilde{\cap}\left(\hbar_{f\mu}|_3,M,N\right);\end{aligned}$$

(5) $$\begin{aligned}&\left(\hbar_{f\mu}|_1,T,N\right)\tilde{\cap}\left(\left(\hbar_{f\mu}|_2,S,N\right)\tilde{\cup}\left(\hbar_{f\mu}|_3,M,N\right)\right)\\&=\left(\left(\hbar_{f\mu}|_2,S,N\right)\tilde{\cap}\left(\hbar_{f\mu}|_1,T,N\right)\right)\tilde{\cup}\begin{pmatrix}\left(\hbar_{f\mu}|_1,T,N\right)\\\tilde{\cap}\left(\hbar_{f\mu}|_3,M,N\right)\end{pmatrix};\end{aligned}$$

(6) $$\begin{aligned}&\left(\hbar_{f\mu}|_1,T,N\right)\tilde{\cup}\left(\left(\hbar_{f\mu}|_2,S,N\right)\tilde{\cap}\left(\hbar_{f\mu}|_3,M,N\right)\right)\\&=\left(\left(\hbar_{f\mu}|_2,S,N\right)\tilde{\cup}\left(\hbar_{f\mu}|_1,T,N\right)\right)\tilde{\cap}\begin{pmatrix}\left(\hbar_{f\mu}|_1,T,N\right)\\\tilde{\cup}\left(\hbar_{f\mu}|_3,M,N\right)\end{pmatrix}\end{aligned}$$

Definition 3.8 Let $\left(\hbar_{f\mu}|_1,T,N_1\right)$ and $\left(\hbar_{f\mu}|_2,S,N_2\right)$ be two *PHFNSSs*, then "$\left(\hbar_{f\mu}|_1,T,N_1\right)$ AND $\left(\hbar_{f\mu}|_2,S,N_2\right)$" denoted by

$$\left(\hbar_{f\mu}|_1,T,N_1\right)\wedge\left(\hbar_{f\mu}|_2,S,N_2\right)=\left(\lambda\!\!\!^{-}_{f_\mu}=\min\left(\hbar_{f_\mu}|_1,\hbar_{f_\mu}|_2\right),T\times S,\min\left(N_1,N_2\right)\right).$$

Definition 3.9 Let $\left(\hbar_{f\mu}|_1,T,N_1\right)$ and $\left(\hbar_{f\mu}|_2,S,N_2\right)$ be two *PHFNSSs*,

then "$\left(\hbar_{f\mu}|_1,T,N_1\right)$ OR $\left(\hbar_{f\mu}|_2,S,N_2\right)$" denoted by

$$\left(\hbar_{f\mu}|_1,T,N_1\right)\vee\left(\hbar_{f\mu}|_2,S,N_2\right)=\left(\lambda\!\!\!^{-}_{f_\mu}=\max\left(\hbar_{f_\mu}|_1,\hbar_{f_\mu}|_2\right),T\times S,\max\left(N_1,\ N_2\right)\right).$$

Example 3.5 Let $\left(\hbar_{f\mu}|_2,S,3\right)$ be another *PHF3SS*, $S=\{e_1,e_2\}$, then

$\left(\hbar_{f\mu}|_2,S,3\right)$ can be obtained as follows:

$$\hbar_{f_\mu}|_2\left(e_1\right)=\left\{\begin{matrix}\left(\dfrac{(u_1,1)}{\{0.20,0.30\}},0.30\right)\\\left(\dfrac{(u_2,2)}{\{0.30,0.40\}},0.30\right)\\\left(\dfrac{(u_3,0)}{\{0.10,0.20,0.30\}},0.20\right)\end{matrix}\right\};$$

$$\hbar_{f_\mu}|_2\left(e_2\right)=\left\{\begin{matrix}\left(\dfrac{(u_1,2)}{\{0.70\}},0.70\right)\\\left(\dfrac{(u_2,0)}{\{0.40,0.60\}},0.40\right)\\\left(\dfrac{(u_3,1)}{\{0.50,0.60\}},0.40\right)\end{matrix}\right\}.$$

Thus,

$$\left(\hbar_{f\mu}|_1,T,N_1\right)\wedge\left(\hbar_{f\mu}|_2,S,N_2\right)=\left(\lambda\!\!\!^{-}_{f_\mu}=\min\left(\hbar_{f_\mu}|_1,\hbar_{f_\mu}|_2\right),T\times S,\min\left(N_1,\ N_2\right)\right).$$

$$\lambda\!\!\!^{-}_{f_\mu}\left(e_1,e_1\right)=\left\{\begin{matrix}\left(\dfrac{(u_1,1)}{\{0.20,0.30\}},0.30\right)\\\left(\dfrac{(u_2,2)}{\{0.30,0.40\}},0.30\right)\\\left(\dfrac{(u_3,0)}{\{0.10,0.20,0.30\}},0.20\right)\end{matrix}\right\};$$

$$\lambda\!\!\!^{-}_{f_\mu}\left(e_1,e_2\right)=\left\{\begin{matrix}\left(\dfrac{(u_1,1)}{\{0.30,0.40\}},0.35\right)\\\left(\dfrac{(u_2,0)}{\{0.30,0.40,0.50\}},0.40\right)\\\left(\dfrac{(u_3,0)}{\{0.30\}},0.30\right)\end{matrix}\right\};$$

$$\lambda\!\!\!^{-}_{f_\mu}\left(e_2,e_1\right)=\left\{\begin{matrix}\left(\dfrac{(u_1,1)}{\{0.20,0.30\}},0.30\right)\\\left(\dfrac{(u_2,0)}{\{0.30,0.40\}},0.30\right)\\\left(\dfrac{(u_3,0)}{\{0.10,0.20,0.30\}},0.20\right)\end{matrix}\right\};$$

$$\lambda\!\!\!^{-}_{f_\mu}\left(e_2,e_2\right)=\left\{\begin{matrix}\left(\dfrac{(u_1,2)}{\{0.70\}},0.70\right)\\\left(\dfrac{(u_2,0)}{\{0.40,0.60\}},0.40\right)\\\left(\dfrac{(u_3,1)}{\{0.50,0.60\}},0.40\right)\end{matrix}\right\};$$

$$ƛ_{f_\mu}(e_3,e_1)=\left\{\begin{array}{l}\left(\dfrac{(u_1,1)}{\{0.10,0.15,0.20\}},0.30\right)\\\left(\dfrac{(u_2,1)}{\{0.10,0.20\}},0.20\right)\\\left(\dfrac{(u_3,0)}{\{0.10\}},0.10\right)\end{array}\right\};$$

$$ƛ_{f_\mu}(e_3,e_2)=\left\{\begin{array}{l}\left(\dfrac{(u_1,1)}{\{0.10,0.15,0.20\}},0.30\right)\\\left(\dfrac{(u_2,0)}{\{0.10,0.20\}},0.20\right)\\\left(\dfrac{(u_3,0)}{\{0.10\}},0.10\right)\end{array}\right\}.$$

Example 3.6 Let $(\hbar_{f\mu}|_2,S,3)$ be another *PHF3SS*, $S=\{e_1,e_2\}$,then $(\hbar_{f\mu}|_1,T,N_1)\vee(\hbar_{f\mu}|_2,S,N_2)$ can be obtained as follows:

$$ƛ_{f_\mu}(e_1,e_1)=\left\{\begin{array}{l}\left(\dfrac{(u_1,1)}{\{0.30,0.40\}},0.35\right)\\\left(\dfrac{(u_2,2)}{\{0.30,0.40,0.50\}},0.40\right)\\\left(\dfrac{(u_3,0)}{\{0.30\}},0.30\right)\end{array}\right\};$$

$$ƛ_{f_\mu}(e_1,e_2)=\left\{\begin{array}{l}\left(\dfrac{(u_1,2)}{\{0.70\}},0.70\right)\\\left(\dfrac{(u_2,2)}{\{0.40,0.60\}},0.40\right)\\\left(\dfrac{(u_3,1)}{\{0.50,0.60\}},0.40\right)\end{array}\right\};$$

$$ƛ_{f_\mu}(e_2,e_1)=\left\{\begin{array}{l}\left(\dfrac{(u_1,2)}{\{0.70,0.80\}},0.75\right)\\\left(\dfrac{(u_2,2)}{\{0.40,0.60\}},0.50\right)\\\left(\dfrac{(u_3,1)}{\{0.30,0.50,0.80\}},0.50\right)\end{array}\right\};$$

$$ƛ_{f_\mu}(e_2,e_2)=\left\{\begin{array}{l}\left(\dfrac{(u_1,2)}{\{0.70,0.80\}},0.75\right)\\\left(\dfrac{(u_2,0)}{\{0.40,0.60\}},0.50\right)\\\left(\dfrac{(u_3,1)}{\{0.30,0.50,0.80\}},0.50\right)\end{array}\right\};$$

$$ƛ_{f_\mu}(e_3,e_1)=\left\{\begin{array}{l}\left(\dfrac{(u_1,1)}{\{0.20,0.30\}},0.30\right)\\\left(\dfrac{(u_2,2)}{\{0.30,0.40\}},0.30\right)\\\left(\dfrac{(u_3,0)}{\{0.10,0.20,0.30\}},0.20\right)\end{array}\right\};$$

$$ƛ_{f_\mu}(e_3,e_2)=\left\{\begin{array}{l}\left(\dfrac{(u_1,2)}{\{0.70\}},0.70\right)\\\left(\dfrac{(u_2,1)}{\{0.40,0.60\}},0.40\right)\\\left(\dfrac{(u_3,1)}{\{0.50,0.60\}},0.40\right)\end{array}\right\}.$$

Thus,

$$(\hbar_{f\mu}|_1,T,N_1)\vee(\hbar_{f\mu}|_2,S,N_2)=$$
$$\left(ƛ_{f_\mu}=\max\left(\hbar_{f_\mu}|_1,\hbar_{f_\mu}|_2\right),T\times S,\max(N_1,N_2)\right).$$

4. APPLICATION

TOPSIS holds for “Technique for Order Preference by Similarity to an Ideal Solution ”. We extend this method TOPSIS (Akram M et al., 2018) so that it can operate under *PHFNSSs* information. From the example 3.1, there are three suppliers $U=\{u_1,u_2,u_3\}$, $T=\{e_1,e_2,e_3\}$ and the parameters stand for budget price, construction quality, construction time. The process is as follows:

Step 1: The positive solution and negative solution are calculated by (1).

$$\hbar^+_{f_\mu}(e_j)=\left\{\begin{array}{l}\left(\dfrac{(u_1,g_1)}{\max\{\lambda^{r_1}{}_{1j}\}},\mu_{1j}\right)\\\cdots\\\left(\dfrac{(u_i,g_i)}{\max\{\lambda^{r_i}{}_{2j}\}},\mu_{ij}\right)\\\cdots\\\left(\dfrac{(u_n,g_n)}{\max\{\lambda^{r_n}{}_{nj}\}},\mu_{nj}\right)\end{array}\right\};\tag{1}$$

$$\hbar^-_{f_\mu}(e_j)=\left\{\begin{array}{l}\left(\dfrac{(u_1,g_1)}{\min\{\lambda^{r_1}{}_{1j}\}},\mu_{1j}\right)\\\cdots\\\left(\dfrac{(u_i,g_i)}{\min\{\lambda^{r_i}{}_{2j}\}},\mu_{ij}\right)\\\cdots\\\left(\dfrac{(u_n,g_n)}{\min\{\lambda^{r_j}{}_{nj}\}},\mu_{nj}\right)\end{array}\right\}$$

Step 2: In order to compute the separation measures H_i^+, H_i^- of each alternative, we calculate the distance between the different attributes as follows: $\forall u_i \in U$,

$$H_i^+ = \left(\alpha_i^+, \beta_{ijl}^{\ +}, \gamma_i^+\right), H_i^- = \left(\alpha_i^-, \beta_{\mathrm{ijl}}^-, \gamma_i^-\right).$$

$$\alpha_i^+ = 2\sum_{j}^{m}\left|g_{ij} - \max\left(g_i\right)\right|,$$

$$\beta_{ijl}^{\ +} = \sqrt{\frac{1}{m^2-m}\sum_{i}^{n}\sum_{l}^{n}\left|\max\left(\lambda^{r_i}{}_{ij}\right) - \max\left(\lambda^{r_i}{}_{lj}\right)\right|^2}, \quad (2)$$

$$\gamma_j^+ = \sum_{i}^{n}\left|\mu_{ij} - \max\left(\mu_j\right)\right|.$$

$$\alpha_i^- = 2\sum_{j}^{m}\left|g_{ij} - \min\left(g_i\right)\right|,$$

$$\beta_{ijl}^{\ -} = \sqrt{\frac{1}{m^2-m}\sum_{j}^{m}\sum_{l}^{m}\left|\min\left(\lambda^{r_i}{}_{ij}\right) - \min\left(\lambda^{r_i}{}_{il}\right)\right|^2}, \quad (3)$$

$$\gamma_j^- = \sum_{i}^{n}\left|\mu_{ij} - \min\left(\mu_j\right)\right|.$$

Step 3: The relative closeness coefficient of each alternative u_i can be computed by using the Equation (4).

$$\wp_j = \frac{H_j^+}{H_j^+ + H_j^-} = \left(\frac{\alpha_j^+}{\alpha_j^+ + \alpha_j^-}, \frac{\beta_{ijl}^+}{\beta_{ijl}^+ + \beta_{ijl}^+}, \frac{\gamma_j^+}{\gamma_j^+ + \gamma_j^-}\right). \quad (4)$$

So according to score of the $\wp_j$, we can determine the ranking order of u_i and choose the best one. We make a decision based on $\left(\hbar_{f\mu}|_1, T, 3\right)$:

Step1: The positive solution and negative solution are calculated by (1).

$$\hbar^+{}_{f_\mu}|_1\left(e_1\right) = \left\{\left(\frac{\left(u_1,1\right)}{0.40}, 0.35\right), \left(\frac{\left(u_2,2\right)}{0.50}, 0.40\right), \left(\frac{\left(u_n,0\right)}{0.30}, 0.30\right)\right\};$$

$$\hbar^+{}_{f_\mu}|_1\left(e_2\right) = \left\{\left(\frac{\left(u_1,2\right)}{0.80}, 0.75\right), \left(\frac{\left(u_2,0\right)}{0.60}, 0.50\right), \left(\frac{\left(u_n,1\right)}{0.80}, 0.50\right)\right\};$$

$$\hbar^+{}_{f_\mu}|_1\left(e_3\right) = \left\{\left(\frac{\left(u_1,1\right)}{0.20}, 0.30\right), \left(\frac{\left(u_2,1\right)}{0.20}, 0.20\right), \left(\frac{\left(u_n,0\right)}{0.10}, 0.10\right)\right\}.$$

Similarly, we can get $\hbar^-_{f_\mu}|_1$.

$$\hbar^-{}_{f_\mu}|_1\left(e_1\right) = \left\{\left(\frac{\left(u_1,1\right)}{0.30}, 0.35\right), \left(\frac{\left(u_2,2\right)}{0.30}, 0.40\right), \left(\frac{\left(u_n,0\right)}{0.30}, 0.20\right)\right\};$$

$$\hbar^-{}_{f_\mu}|_1\left(e_2\right) = \left\{\left(\frac{\left(u_1,2\right)}{0.70}, 0.75\right), \left(\frac{\left(u_2,0\right)}{0.40}, 0.50\right), \left(\frac{\left(u_n,1\right)}{0.30}, 0.50\right)\right\};$$

$$\hbar^-{}_{f_\mu}|_1\left(e_3\right) = \left\{\left(\frac{\left(u_1,1\right)}{0.10}, 0.30\right), \left(\frac{\left(u_2,1\right)}{0.10}, 0.20\right), \left(\frac{\left(u_n,0\right)}{0.10}, 0.10\right)\right\}.$$

Step 2: According to the equation (2) (3), we have

$$\begin{array}{lll} \alpha_1^+ = 4 & \beta_{1jl}^{\ +} = 0.43 & \lambda_1^+ = 0.85 \\ \alpha_2^+ = 6 & \beta_{2jl}^{\ +} = 0.29 & \lambda_2^+ = 0.40 \\ \alpha_3^+ = 4 & \beta_{3jl}^{\ +} = 0.51 & \lambda_3^+ = 0.70 \\ \alpha_1^- = 2 & \beta_{1jl}^{\ -} = 0.43 & \lambda_1^- = 0.50 \\ \alpha_2^- = 4 & \beta_{2jl}^{\ -} = 0.22 & \lambda_2^- = 0.50 \\ \alpha_3^- = 2 & \beta_{3jl}^{\ -} = 0.16 & \lambda_3^- = 0.50 \end{array}$$

Step 3: Calculating $\wp_i$ from the equation (4):

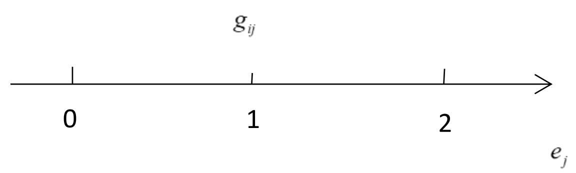
$$\wp_1 = \left(0.67, 0.50, 0.63\right);$$
$$\wp_2 = \left(0.60, 0.57, 0.44\right);$$
$$\wp_3 = \left(0.67, 0.76, 0.58\right).$$

For the $\wp_i$: we get a rank:

$$u_3 = 0.67 = u_1 = 0.67 > u_2 = 0.60$$
$$u_3 = 0.67 > u_2 = 0.57 > u_1 = 0.50$$
$$u_3 = 0.58 > u_1 = 0.63 > u_2 = 0.44.$$

Obviously, $u_3 > u_1 > u_2$. Supplier u_3 can win the bid.

In this application, we determine the grade on a per-attribute basis so the grade determination needs to be consistent.



Thus the bigger the grades means the better the quality, the more time and the higher the budget. The smaller grades mean lower quality, less time and lower budget from this number line.

5 Conclusion

This paper defined a novel structure called possibility hesitant fuzzy *N*-Soft sets which demonstrates the not only the degree of belongingness of the elements of *U*, but also the degree of possibility of possibility of such belongingness. The grades are in the gift of hesitant fuzzy element and the number of elements. This application fits the reality, when bidding, companies hope to find suppliers with good quality, low budget price and short construction time. For the properties e_3 = construction time, the lower the grades, the shorter the time. For e_2 = construction quality, the higher the grades, the better the quality. Finally, we extend this method TOPSIS so that it can operate under *PHFNSSs* information.

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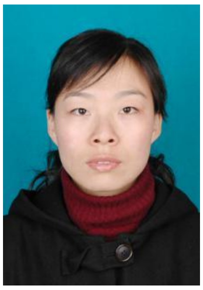

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