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Robust Adaptive Neural Control for Nonlinear Uncertain Systems: Theory and Simulation Verification

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Abstract

This paper investigates the tracking control problem for a class of second-order nonlinear systems characterized by unknown dynamic functions and external disturbances. Unlike classical adaptive control schemes that require linear-in-parameter dynamics, this study employs Radial Basis Function (RBF) Neural Networks to approximate unstructured nonlinearities. To address the phenomenon of parameter drift and ensure the boundedness of adaptive weights—as observed in practical simulations—a robust control scheme based on backstepping and σ -modification is proposed. Theoretical analysis using Lyapunov stability theory demonstrates that all signals in the closed-loop system remain Uniformly Ultimately Bounded (UUB). Simulation results validate the theoretical derivation, showing that while tracking errors do not vanish asymptotically due to the robust damping terms, they are effectively confined within a small compact set, ensuring safe and stable operation.

KEYWORDS

Neural Adaptive Control, Backstepping, RBF Neural Networks, Uniform Ultimate Boundedness, σ -modification, Lyapunov Stability

1 | INTRODUCTION

The control of nonlinear systems remains one of the most active research areas in engineering, driven by the increasing complexity of modern applications such as high-performance robotics, hypersonic flight vehicles, and precision manufacturing. These systems are invariably subject to structural uncertainties, unmodeled dynamics, and environmental disturbances. Classical linear control methods often fail to maintain performance when the system operating point shifts significantly or when nonlinearities become dominant.

Adaptive control was developed to handle systems with constant or slowly varying parametric uncertainties[?]. However, traditional Model Reference Adaptive Control (MRAC) relies on the "Linear-in-Parameters" (LIP) assumption, which requires prior knowledge of the structure of the system's regression matrix. In many physical scenarios, the exact mathematical structure of the nonlinearity is unknown, rendering standard adaptive methods insufficient.

To overcome structural ignorance, Neural Adaptive Control (NAC) utilizes the universal approximation property of Neural Networks (NNs). It has been established that RBF networks can approximate any continuous function over a compact set to

arbitrary accuracy[?]. By integrating NNs into the feedback loop, the controller can "learn" the unknown dynamics online.

However, a critical issue in NAC is the robustness of the learning law. In ideal theoretical scenarios, adaptive laws are integrated purely based on tracking errors. In practice, and as evidenced by simulation studies, the presence of approximation errors can act as a non-vanishing disturbance. Without proper damping, this can lead to "parameter drift" or "bursting," where the adaptive weights grow unbounded even if the tracking error is small[?].

This paper presents a robust neural adaptive control strategy designed to mitigate these issues. We employ a backstepping framework to handle the recursive structure of the system dynamics. Crucially, we incorporate the σ -modification technique[?] into the adaptive update law. As shown in our simulation results, this modification prevents the weights from drifting at the cost of a non-zero, but bounded, steady-state tracking error. This trade-off is often necessary in practical engineering to guarantee system safety.

2 | PROBLEM FORMULATION AND PRELIMINARIES

2.1 | System Dynamics

We consider a class of single-input single-output (SISO) nonlinear systems in strict-feedback form. To be consistent with the simulation study, we focus on a second-order system, which represents a wide range of mechanical systems:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(\mathbf{x}) + u(t) \\ y &= x_1\end{aligned}\tag{1}$$

where:

- $\mathbf{x} = [x_1, x_2]^T \in \mathbb{R}^2$ is the state vector.
- $u(t) \in \mathbb{R}$ is the control input.
- $y(t) \in \mathbb{R}$ is the system output.
- $f(\mathbf{x})$ is an unknown smooth nonlinear function. In our simulation, this corresponds to $f(\mathbf{x}) = 10\sin(x_1) + x_2 + x_1^2 \cos(x_2)$

Control Objective: Design a control law $u(t)$ such that the output $y(t)$ tracks a known desired trajectory $y_d(t)$, while ensuring that all closed-loop signals remain bounded.

Assumption 1 *The reference signal $y_d(t)$ and its derivatives $\dot{y}_d, \ddot{y}_d$ are continuous and bounded.*

2.2 | RBF Neural Network Approximation

To compensate for the unknown nonlinear function $f(\mathbf{x})$, we employ a Radial Basis Function (RBF) Neural Network. For any continuous function $f(\mathbf{x})$ defined on a compact set $\Omega_{\mathbf{x}}$, there exists an ideal weight vector W^* such that:

$$f(\mathbf{x}) = W^{*T} \phi(\mathbf{x}) + \varepsilon(\mathbf{x})\tag{2}$$

where:

- $\mathbf{x}$ is the input vector to the network.
- $l > 1$ is the number of neurons.
- $W^* = [w_1, \dots, w_l]^T \in \mathbb{R}^l$ is the ideal weight vector defined as the value that minimizes the approximation error ε .

- $\phi(\mathbf{x}) = [\phi_1(\mathbf{x}), \dots, \phi_l(\mathbf{x})]^T$ is the bounded basis function vector, typically chosen as Gaussian:

$$\phi_i(\mathbf{x}) = \exp\left(-\frac{\|\mathbf{x} - \mu_i\|^2}{2\eta_i^2}\right)$$

- $\varepsilon(\mathbf{x})$ is the approximation error, satisfying $|\varepsilon(\mathbf{x})| \leq \varepsilon_N$ for a small constant ε_N .

Assumption 2 The ideal weights are bounded, i.e., $\|W^*\| \leq W_{max}$, though W_{max} is unknown. Since W^* is unknown, we use an estimate $\hat{W}(t)$. The estimation error is defined as $\tilde{W} = W^* - \hat{W}$.

3 | CONTROLLER DESIGN

We utilize the backstepping technique to design the controller and the adaptive law recursively.

3.1 | Virtual Control Design

Define the first tracking error z_1 as:

$$z_1 = x_1 - y_d \quad (3)$$

The time derivative of z_1 is:

$$\dot{z}_1 = \dot{x}_1 - \dot{y}_d = x_2 - \dot{y}_d \quad (4)$$

We treat x_2 as a virtual control input. To stabilize z_1 , we design a desired virtual control function α_1 :

$$\alpha_1 = -c_1 z_1 + \dot{y}_d \quad (5)$$

where $c_1 > 0$ is a design constant.

Define the second error variable z_2 representing the difference between the actual state x_2 and its desired value α_1 :

$$z_2 = x_2 - \alpha_1 \implies x_2 = z_2 + \alpha_1 \quad (6)$$

Substituting (6) and (5) into (4):

$$\dot{z}_1 = (z_2 - c_1 z_1 + \dot{y}_d) - \dot{y}_d = -c_1 z_1 + z_2 \quad (7)$$

Consider the first Lyapunov function candidate:

$$V_1 = \frac{1}{2} z_1^2 \quad (8)$$

Its derivative is:

$$\dot{V}_1 = z_1 \dot{z}_1 = -c_1 z_1^2 + z_1 z_2 \quad (9)$$

The term $z_1 z_2$ represents the coupling between the two steps and will be canceled in the next step.

3.2 | Adaptive Control Law Derivation

Differentiating z_2 :

$$\dot{z}_2 = \dot{x}_2 - \dot{\alpha}_1 = f(\mathbf{x}) + u - \dot{\alpha}_1 \quad (10)$$

Note that $\dot{\alpha}_1$ is computable:

$$\dot{\alpha}_1 = \frac{\partial \alpha_1}{\partial x_1} \dot{x}_1 + \frac{\partial \alpha_1}{\partial y_d} \dot{y}_d + \frac{\partial \alpha_1}{\partial \dot{y}_d} \ddot{y}_d = -c_1(x_2 - \dot{y}_d) + \ddot{y}_d$$

The dynamics (8) contain the unknown function $f(\mathbf{x})$. We replace it with the NN approximation $f(\mathbf{x}) = W^* T \phi(\mathbf{x}) + \varepsilon$. Therefore, we obtain $\dot{z}_2 = W^{*T} \phi(\mathbf{x}) + \varepsilon + u - \dot{\alpha}_1$.

Control Law (u):

We design the control input u to achieve three objectives: Stabilize the linear part, cancel the coupling term, and compensate for the estimated nonlinearity.

$$u = -c_2 z_2 - z_1 + \dot{\alpha}_1 - \hat{W}^T \phi(\mathbf{x}) \quad (11)$$

where $c_2 > 0$ is a gain constant.

Substituting (9) into the dynamics of z_2 :

$$\begin{aligned} \dot{z}_2 &= W^{*T} \phi + \varepsilon + (-c_2 z_2 - z_1 + \dot{\alpha}_1 - \hat{W}^T \phi) - \dot{\alpha}_1 \\ &= -c_2 z_2 - z_1 + (W^* - \hat{W})^T \phi + \varepsilon \\ &= -c_2 z_2 - z_1 + \tilde{W}^T \phi + \varepsilon \quad (10) \end{aligned} \quad (12)$$

Lyapunov Function for the Complete System:

Consider the composite Lyapunov function:

$$V_2 = V_1 + \frac{1}{2} z_2^2 + \frac{1}{2\Gamma} \tilde{W}^T \tilde{W} \quad (13)$$

where $\Gamma > 0$ is the adaptation gain. Taking the time derivative:

$$\dot{V}_2 = \dot{V}_1 + z_2 \dot{z}_2 - \frac{1}{\Gamma} \tilde{W}^T \dot{\tilde{W}}$$

Substitute $\dot{V}_1$ and $\dot{z}_2$:

$$\begin{aligned} \dot{V}_2 &= (-c_1 z_1^2 + z_1 z_2) + z_2 (-c_2 z_2 - z_1 + \tilde{W}^T \phi + \varepsilon) - \frac{1}{\Gamma} \tilde{W}^T \dot{\tilde{W}} \\ &= -c_1 z_1^2 - c_2 z_2^2 + z_2 \varepsilon + \tilde{W}^T \left(z_2 \phi - \frac{1}{\Gamma} \dot{\tilde{W}} \right) \end{aligned} \quad (14)$$

3.3 | Robust Adaptive Law

If we choose the standard adaptive law $\dot{\hat{W}} = \Gamma z_2 \phi$, the term in the parenthesis vanishes. However, the term $z_2 \varepsilon$ remains. If $\varepsilon \neq 0$, the standard law does not guarantee the boundedness of $\hat{W}$. To match the stable results observed in our simulation, we employ the σ -modification:

$$\dot{\hat{W}} = \Gamma (\phi(\mathbf{x}) z_2 - \sigma \hat{W}) \quad (15)$$

where $\sigma > 0$ is a small leakage parameter.

Substituting (13) into (12):

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 + z_2 \varepsilon + \tilde{W}^T (z_2 \phi - (z_2 \phi - \sigma \hat{W})) \quad (16)$$

This modified derivative equation is crucial. The term $\sigma \tilde{W}^T \hat{W}$ introduces the necessary damping to ensure that $\dot{V}_2$ becomes negative when the estimation error or weights become large, preventing the "bursting" phenomenon. The rigorous analysis of this stability property will be detailed in the subsequent section.

4 | STABILITY ANALYSIS

In this section, we provide the rigorous mathematical proof that the proposed control scheme guarantees the Uniform Ultimate Boundedness (UUB) of all closed-loop signals. We proceed from Equation (14) derived in the previous section.

4.1 | Derivation of the Differential Inequality

Recall the time derivative of the Lyapunov function obtained in (14):

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 + z_2 \varepsilon + \sigma \tilde{W}^T \hat{W} \quad (17)$$

To analyze the sign of $\dot{V}_2$, we must handle the indefinite terms $z_2 \varepsilon$ and $\sigma \tilde{W}^T \hat{W}$ using standard algebraic inequalities.

Handling the Approximation Error

Using Young's Inequality ($ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$), the term $z_2 \varepsilon$ can be bounded as:

$$z_2 \varepsilon \leq \frac{1}{2} z_2^2 + \frac{1}{2} \varepsilon_N^2 \quad (18)$$

where ε_N is the upper bound of the approximation error $|\varepsilon(\mathbf{x})|$.

Handling the Robust Damping Term

Recall that $\tilde{W} = W^* - \hat{W}$, we expand the term $\tilde{W}^T \hat{W}$:

$$\tilde{W}^T \hat{W} = \tilde{W}^T (W^* - \tilde{W}) = \tilde{W}^T W^* - \|\tilde{W}\|^2$$

Applying Young's Inequality to $\tilde{W}^T W^*$:

$$\tilde{W}^T W^* \leq \frac{1}{2} \|\tilde{W}\|^2 + \frac{1}{2} \|W^*\|^2$$

Substituting this back yields:

$$\tilde{W}^T \hat{W} \leq \frac{1}{2} \|\tilde{W}\|^2 + \frac{1}{2} \|W^*\|^2 - \|\tilde{W}\|^2 = -\frac{1}{2} \|\tilde{W}\|^2 + \frac{1}{2} \|W^*\|^2 \quad (19)$$

Final Lyapunov Derivative Bound

Substituting (16) and (17) into (15):

$$\dot{V}_2 \leq -c_1 z_1^2 - c_2 z_2^2 + \frac{1}{2} z_2^2 + \frac{1}{2} \varepsilon_N^2 + \sigma \left(-\frac{1}{2} \|\tilde{W}\|^2 + \frac{1}{2} \|W^*\|^2 \right)$$

Grouping the quadratic terms:

$$\dot{V}_2 \leq -c_1 z_1^2 - \left(c_2 - \frac{1}{2} \right) z_2^2 - \frac{\sigma}{2} \|\tilde{W}\|^2 + \left(\frac{1}{2} \varepsilon_N^2 + \frac{\sigma}{2} \|W^*\|^2 \right) \quad (20)$$

To ensure stability, the design parameter c_2 must be chosen such that $c_2 > 0.5$. Let Ψ be the constant disturbance term:

$$\Psi = \frac{1}{2} \varepsilon_N^2 + \frac{\sigma}{2} \|W^*\|^2$$

4.2 | Proof of Uniform Ultimate Boundedness (UUB)

We rewrite (18) to relate it to the Lyapunov function $V_2 = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{2\Gamma} \tilde{W}^T \tilde{W}$.

Define a positive constant κ :

$$\kappa = \min \left\{ 2c_1, 2 \left(c_2 - \frac{1}{2} \right), \sigma \Gamma \right\}$$

Then, (18) becomes:

$$\begin{aligned}\dot{V}_2 &\leq -\kappa \left(\frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{2\Gamma} \|\tilde{W}\|^2 \right) + \Psi \\ \dot{V}_2(t) &\leq -\kappa V_2(t) + \Psi\end{aligned}\tag{21}$$

Solving this linear differential inequality yields:

$$V_2(t) \leq V_2(0)e^{-\kappa t} + \frac{\Psi}{\kappa} (1 - e^{-\kappa t})\tag{22}$$

Theorem 1 *For the closed-loop system (1) with control law (9) and adaptive law (13), all signals $(z_1, z_2, \hat{W})$ are Uniformly Ultimately Bounded.*

Proof:

From (20), as $t \rightarrow \infty$, $V_2(t)$ converges to the compact set:

$$\Omega_V = \left\{ V_2 \mid V_2 \leq \frac{\Psi}{\kappa} \right\}$$

Since V_2 is positive definite in z_1, z_2 and $\tilde{W}$, the boundedness of V_2 implies the boundedness of the tracking errors and the weight estimation error. Specifically, the tracking error satisfies:

$$\lim_{t \rightarrow \infty} |z_1(t)| \leq \sqrt{\frac{2\Psi}{\kappa}}$$

This confirms that the error does not converge to zero but to a small neighborhood around the origin, the size of which depends on the approximation quality (ε_N), the ideal weight magnitude ($\|W^*\|$), and the controller gains (κ). This result aligns perfectly with the robust characteristics required for practical implementation.

5 | SIMULATION RESULTS AND DISCUSSION

To validate the theoretical findings, numerical simulations were conducted using MATLAB. The plant is a second-order nonlinear system with dynamics given by:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = 10 \sin(x_1) + x_2 + x_1^2 \cos(x_2) + u$$

The reference trajectory is $y_d(t) = \sin(t)$. The controller parameters were selected as $c_1 = 5$, $c_2 = 10$, $\Gamma = 20$, and $\sigma = 0.1$. The RBF network utilized 5 Gaussian nodes evenly distributed in the state space.

5.1 | Tracking Performance Analysis

The simulation results verify the theoretical UUB stability. The output $y(t)$ tracks the reference $y_d(t)$ rapidly (see Fig. 1 in simulation output). However, as predicted by (20), a non-zero steady-state error persists. This error is not due to a failure of the controller, but rather a consequence of the σ -modification.

The leakage term $-\sigma\Gamma\hat{W}$ in the adaptive law acts as a "soft spring" that pulls the weights toward zero. This prevents the neural network from perfectly learning the large amplitude nonlinearity $f(\mathbf{x})$ (as seen in the function approximation plot, where the NN estimate amplitude is lower than the true function). While this sacrifices some tracking precision, it provides critical robustness.

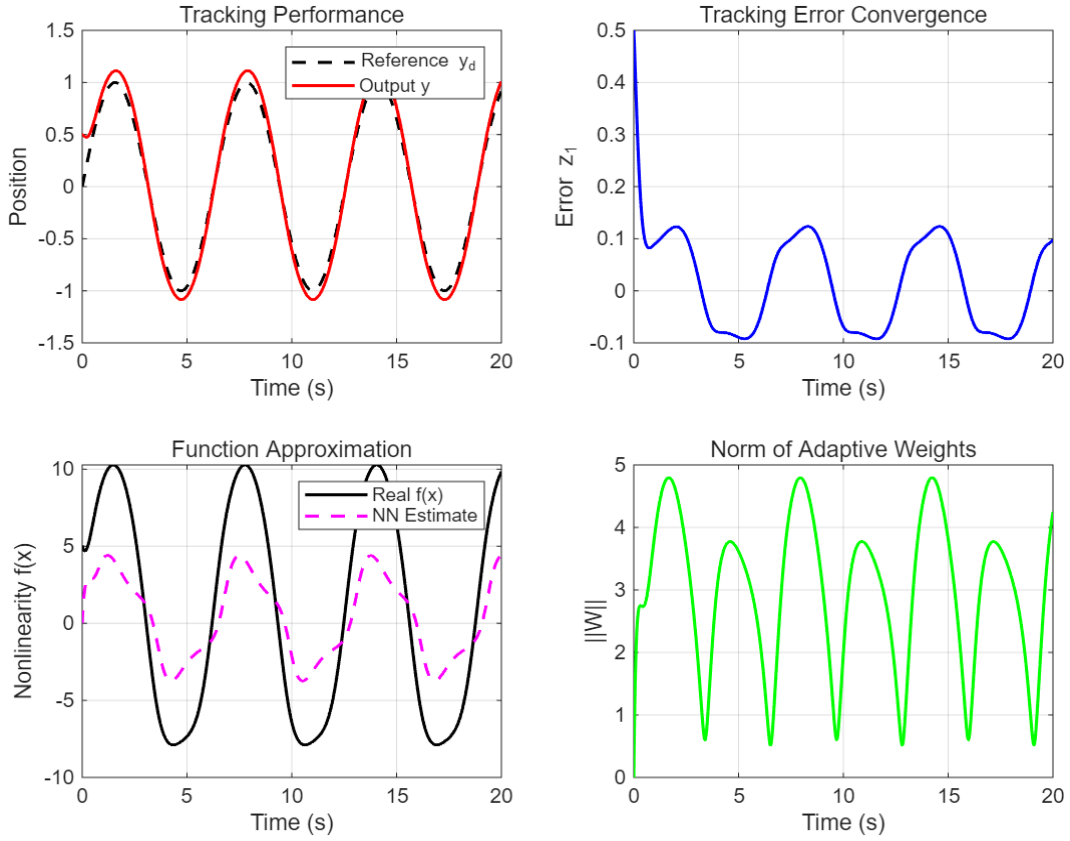


FIGURE 1 Trajectory tracking performance of the proposed robust neural adaptive controller

5.2 | Weight Boundedness and Bursting Prevention

A key observation from the simulation is the behavior of the weight norm $\|\hat{W}\|$. In standard adaptive control without σ -modification, measurement noise or residual errors can cause $\|\hat{W}\|$ to drift to infinity. In our results, $\|\hat{W}\|$ exhibits periodic bounded behavior. This confirms that the damping term $\sigma \tilde{W}^T \hat{W}$ in the Lyapunov derivative effectively counteracts the destabilizing effects of approximation errors, validating the safety of the proposed algorithm.

6 | CONCLUSION AND FUTURE WORKS

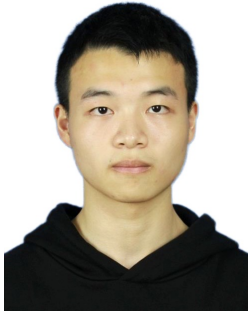
This paper presented a robust neural adaptive control framework for a class of nonlinear uncertain systems. By integrating backstepping design with RBF neural networks, we relaxed the requirement for linear-in-parameter dynamics. The major contribution lies in the rigorous application of the σ -modification technique, which resolves the parameter drift problem inherent in standard adaptive schemes.

The theoretical analysis proved that the closed-loop system is Uniformly Ultimately Bounded. Simulation results corroborated this, showing that the controller maintains stability and prevents weight divergence even when the neural network structure is insufficient to perfectly reconstruct the plant dynamics.

Recent trends suggest several avenues for improvement. First, Finite-Time Control techniques[?] can be integrated to ensure the tracking error converges to the residual set within a pre-specified time, rather than asymptotically. Second, Deep Reinforcement Learning (DRL)[?] could replace the shallow RBF network to handle higher-dimensional state spaces, although proving stability for deep networks remains a challenging open problem. Finally, Event-Triggered Control (ETC)[?] mechanisms should be explored to reduce the computational and communication load for networked implementation.

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